

Example of a final exam in Switzerland

In Switzerland every canton has its own educational system. Therefore no centralised final examinations exist, i.e. each teacher usually arranges his or her own examination. However, all examinations are published to assure quality. The following final exam is merely an example of many variants. Usually mathematics is tested with a written and an oral exam. The following is an example of an exam without an oral test. In lieu of the oral exam, there is a first part consisting of simpler traditional tasks that are completed without use of a calculator. For the sake of completeness this first part is included as well (without solutions).

1. 1st part: Without calculator - Time 90 minutes

All of category C0 or C1

- (1) A sphere with centre $M = (2; 4; 0)$ has the point $A = (4; 5; -2)$ on its surface.
 - a) Determine the equation of the sphere.
 - b) B is the point on the spherical surface which lies nearest to the plane $E: x + 2y + 2z + 26 = 0$
Calculate the distance from B to E .
- (2) A circle is inscribed into a square with sides a . Now a square is inscribed into this circle etc. Determine the (infinite) sum of the areas of all circles.
- (3) $f_n(x) = n^2 \cdot x^3 - n \cdot x^2 + x$ ($n \in \mathbb{N}$)
 - a) Show that all inflection points W_n lie on a straight line.
Determine the equation of this straight line.
 - b) Examine whether the tangents of inflection are parallel, coincide or intersect.
- (4) Determine the area limited by the y -axis and the curves $y = \sin x$, $x \geq 0$ and $y = \cos x$, $x \geq 0$ (up to the first intersection point).
- (5) A ray of light coming from the point $A = (0; -2; 1)$ has the direction $\vec{a} = \begin{pmatrix} 4 \\ 8 \\ 0 \end{pmatrix}$ and is reflected at the plane $E: x + 3y + 2z - 24 = 0$.
 - a) Determine the point of reflection X .
 - b) Determine a parametric representation of the reflected ray of light.
- (6) A student receives a loan L . It is to be paid back in 8 equal annual payments and the first repayment is made 6 years after receipt of the loan. The interest rate is p %.
How large is a repayment?

2. 2nd part: With CAS-Calculator - Time: 120 minutes

The first and second exercises are classical tasks. They could also be done by-hand, but it would be a time consuming effort. Exercises 3 and 4 require the use of a calculator. We will only give the solutions for these exercises. Since the development of such examination tasks is very important, we will show you the “history” of the development of exercise 4.

(1) $P = (u; v)$ is an arbitrary point on the parabola with the equation

Category C1

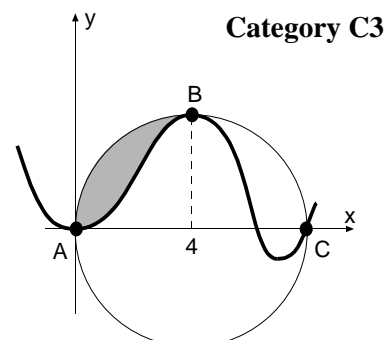
$$y = -\frac{1}{2}x^2 + 2 \quad (-2 \leq x \leq 2).$$

- Determine P such that in the triangle ABP with $A = (-2; 0)$ and $B = (u; 0)$ the sum of the cathetes is maximal.
- By rotating the triangle ABP around the x -axis a cone arises. Which is the maximum volume of such a cone?
- The maximum cone from subtask b) is cut out from the rotation solid of the parabola (rotation around x -axis). What is the percentage of the waste that arises?
- The normal to the parabola cuts the x -axis at point N . Determine P such that the triangle ANP has the greatest possible area. How big is the maximum area?

(2) A roof of a tower with a square base has a shape like that shown in the figure to the right (length unit 1 m).

Category C1

364 TD0.056 0 0 6.900942799 516 such that the triangle



b) Outline the real curve as well as the circle.

Tip: Accelerate the calculation by using a numeric value in the function.

c) Choose a parameter k (instead of 6) for the integral between A and B . Can k be arbitrary? (Give a reason for your answer.) If the answer is yes, outline the curve for $k = 0$ (including the circle).

d) Discuss the asymptotic behaviour for $x \rightarrow \pm\infty$ for allowable values of k .

(4) An aeroplane flies on a greater circle at 10 km altitude around the earth. **Category C3**
It passes Pyongyang in North Korea (126° east; 39° north) and Khartoum in Sudan (33° east; 16° north). Earth radius $r = 6371$ km.

a) The aeroplane needs 20 hours from Pyongyang to Khartoum. What is its average speed?

b) At which point (geographical length) does the plane cross the equator for the first time after flying past Khartoum?

c) What is the northern-most point of the flight path (Cartesian coordinates are sufficient)?

d) Does Iraq lie on this greater circle?

Iraq extends from latitude 27° to 37.5° north on the meridian 44° east through Baghdad.

3. Credits for the exam

First part: 30 points (5 points per task)

Second part: 40 points (10 points per task).

Altogether: 70 points altogether.

Highest mark for 45 points or more. The pass mark was 23 points. Average was 37.3 points.

4. Solution to task 3 (2nd part)

Solution to a) and b):

Establish the conditions by-hand:

$$f(0) = 0, f'(0), f(4) = 4, f'(4), f(8) = 0, f'(8), A = 4\pi - \int_0^4 f(x)dx = 6.$$

Determine the polynomial due to the number of conditions (here 7):

$$f(x) = a \cdot x^6 + b \cdot x^5 + c \cdot x^4 + d \cdot x^3 + e \cdot x^2$$

Please let the students find the degree of the polynomials, because it works in any case, but the result will be wrong and the graph provides them with an immediate self-check.

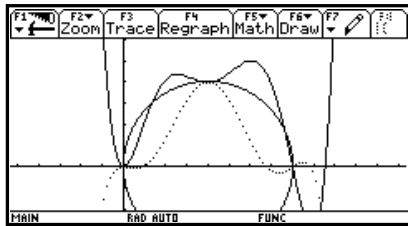
To save calculation time the last two constants were set to zero (according to the first two conditions). Now we determine the first and second derivatives and store them in $f1(x)$ and $f2(x)$. We store the remaining five equations above in e1, e2, e3, e4 and e5 and solve the corresponding system of equations: solve (e1 and e2 and e3 and e4 and e5, {a, b, c, d, e}).

The result appears in the form $a = \dots$ and $b = \dots$ and $\dots e = \dots$ We store $f(x)$ | result in $y1(x)$ and we get (numeric values to accelerate the drawing process)

$$y1(x) = -0.001x^6 + 0.036x^5 - 0.300x^4 + 0.930x^3 - 0.587x^2.$$

Now draw the function and the circle (as a two-part function or approximately with F7 Circle in the graph window, or with Circle 4,0,4 in the Home Screen).

Solution to c):



Modify the last equation and replace 6 by k and solve the system of equations again (all taken from history). We see that in the new function k appears in linear terms only and therefore the system is always solvable.

Store the function in $y2(x)$ and draw it for $k = 0$ (see figure, dotted line for $k = 6$).

Solution to d):

The asymptotic behaviour depends on the sign of the highest coefficient x^6 .

The coefficient of x^6 is $a = -0.000657(k-3.766)$, thus

$$k > 3.766 \Rightarrow \lim_{x \rightarrow \pm\infty} f(x) = -\infty$$

$$k < 3.766 \Rightarrow \lim_{x \rightarrow \pm\infty} f(x) = +\infty$$

Special case: $k = 0 \Rightarrow$ Coefficient of $x^5 < 0 \Rightarrow \lim_{x \rightarrow \infty} f(x) = \infty$ and $\lim_{x \rightarrow -\infty} f(x) = -\infty$

Student solution to task 3

Kantonsschule Heerbrugg
Maturitätsprüfungen
Klasse: M Wa Ta

Name und Vorname: _____
Fach: Mathematik

3)

a) Bedingungen:

1. $f(0) = 0 \checkmark \Rightarrow g = 0$
2. $f'(0) = 0 \checkmark \Rightarrow f = 0$
3. $f(4) = 4 \checkmark$
4. $f'(4) = 0 \checkmark$
5. $f''(8) = 0 \checkmark$
6. $f'''(8) = 0 \checkmark$
7. $\int_0^4 f(x) dx = 6 \checkmark$

\Rightarrow Polynom 6. Grades: $f(x) = ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g$

\Rightarrow TR: $f(x) = -0.001468x^6 + 0.035892x^5 - 0.3x^4 + 0.92x^3 + 0.5873x^2$

b)

c)

$$a = -0.000657(k - 3.7664)$$

$$b = 0.01665(k - 3.84457)$$

$$c = -0.1507(k - 4.00823)$$

$$d = 0.5749(k - 4.3824)$$

$$e = -0.785256(k - 5.25208)$$

Wird k kann zwischen 0 und 4π liegen.

Negativ darf k nicht sein, sonst geht die Kurve nicht durch B!

$k=0$:

d) Parameter a (bei höchster Potenz x^6) $a = -0.000657(k - 3.7664)$

Wenn $k < 3.7664 \Rightarrow x \rightarrow \infty : y \rightarrow -\infty$
 $x \rightarrow -\infty : y \rightarrow \infty$ ✓

Wenn $k > 3.7664 \Rightarrow x \rightarrow \infty : y \rightarrow \infty$
 $x \rightarrow -\infty : y \rightarrow -\infty$ ✓

Wenn $k = 3.7664 \Rightarrow a = 0 \Rightarrow x \rightarrow \infty : y = -\infty$
 $x \rightarrow -\infty : y = \infty$ ✓

9 1/2

Introduction of parameters leads to a new dynamic in algebra. With computer programs you can vary the parameter k with sliders and change the graph of the function interactively. In geometry this interactivity is well known through dynamic geometry programs. Of course, the size of the area is not the only number, which can be parameterised. In each equation numbers can be replaced by parameters. By doing so you get unlimited possibilities for individual and group activities and their presentation in the form of animations.

5. Solution to task 4 (2nd part)

- a) Store the polar coordinates as a (vector) function of three variables, namely radius r , degree of longitude x and degree of latitude y :

$$\vec{v}(r, x, y) = \begin{pmatrix} r \cos(x) \cos(y) \\ r \sin(x) \cos(y) \\ r \sin(y) \end{pmatrix}.$$

With CAS: $[r \cos(x) \cos(y), r \sin(x) \cos(y), r \sin(y)] \text{ [STO]} \vec{v}(r, x, y)$.

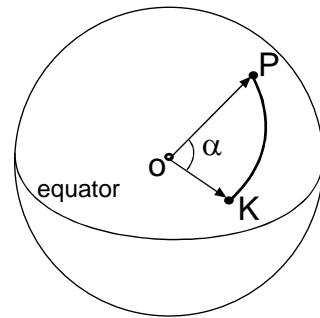
Define the position vectors for Pyongyang P and Khartoum K . To avoid large numbers we take $r = 1$ (unit sphere).

$$\overline{OP} \parallel \vec{p} = v(1, 126^\circ, 39^\circ) \quad \overline{OK} \parallel \vec{k} = v(1, 33^\circ, 16^\circ)$$

Store these vectors as numerical values. Switch now to radian mode and calculate the angle between \vec{p} and \vec{k}

$$\alpha = \arccos \frac{\vec{p} \cdot \vec{k}}{|\vec{p}| |\vec{k}|} = 1.436$$

$$\text{Hence } v = \frac{s}{t} = \frac{6381 \cdot \alpha}{20} = 458.163 \frac{\text{km}}{\text{h}}.$$



- b) *Students think differently when solving this problem with CAS. By-hand students fear the cross product like the devil the holy water, because it requires much calculation. To find a vector for a certain direction you have to find two vectors perpendicular to it and CAS calculates the cross product in no time. Students do the more important mathematical part, CAS the calculation part.*

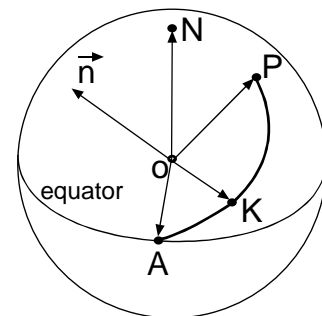
Let $\vec{n} = \overline{OK} \times \overline{OP}$ be a normal vector to the greater circle through Pyongyang and Khartoum and A the intersection point of this circle with the equatorial circle. Then we have the following conditions:

$$\overline{OA} \perp \vec{n} \text{ and } \overline{OA} \perp \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \overline{OA} \parallel \vec{e}_3 \times \vec{n}$$

If we take the unit vector $\vec{e}_A = \frac{\overline{OA}}{|\overline{OA}|}$, its first component will

just be the cosine of the angle sought. Therefore we get the longitude of point A (switch to degree mode):

$$\varphi = \arccos(\text{1st component of } \vec{e}_A) \stackrel{\text{CAS}}{=} \arccos(\vec{e}_A[1,1]) = 13.854^\circ$$



- c) Let H be the highest (northern-most) point of the flight path. We have the following conditions:
 $\overline{OH} \perp \vec{n}$ and $\overline{OH} \perp \overline{OA} = k\vec{e}_A \Rightarrow \vec{h} = \vec{n} \times \vec{e}_A$ is a vector in direction of \overline{OH} .

$$\text{Hence } \overline{OH} = 6381 \frac{\vec{h}}{|\vec{h}|} = \begin{pmatrix} -1150.281 \\ 4664.148 \\ 4199.968 \end{pmatrix}.$$

- d) Store the unit vector $\vec{i} = v(1, 44^\circ, x)$ of the Iraqi meridian.

The condition for the angle x to the greater circle is $\vec{n} \times \vec{i} = 0$.

This equation is solved with respect to x and results in: $x = 180^\circ \cdot (k + 0.1317)$.

For $k = 0$ we get $x = 23.705^\circ$, which means that the greater circle passes south of Iraq.

Using defined functions as a module makes CAS-calculators very efficient and encourages the tackling of realistic problems. Both of the above tasks would be very difficult to solve by-hand or with scientific calculators. Tasks of this kind look very difficult at first glance, but with some training they become standard problems for students.

6. "History" of task 4 (2nd part)

It is certainly of interest to know which exercises and examinations preceded this final exam. Therefore we will list all examples, exercises and exam questions completed in preparation for task 4. The subject was dealt with approximately seven months before the final exam. The solutions are similar to those of task 4 and thus are omitted. The beginning of the instructional unit included the theory of the cross product with some standard examples without use of polar coordinates and often solved by-hand. The following exercises are hardly solvable by-hand. With the time saved CAS students can solve many more problems about the cross product. (On the other hand tasks about intersection of planes and lines become less important.) The main idea is as mentioned above: if we search for a vector, we have to find two vectors perpendicular to this vector.

Exercise 1 (without polar coordinates)

Category C1

Given two vertices $A = (4; 0; 1)$, $B = (0; 2; 7)$ of a tetrahedron $ABCD$ and a point $P = (1; 1; 1)$ in the plane ABC .

- Determine the vertices C and D of a regular tetrahedron $ABCD$. Calculate just one solution and give the number of possible solutions.
- Determine the vertex E of a tetrahedron $ABCE$ with an equilateral triangle ABC such that the height through E passes the centroid of ABC and E lies in the x, y -plane.

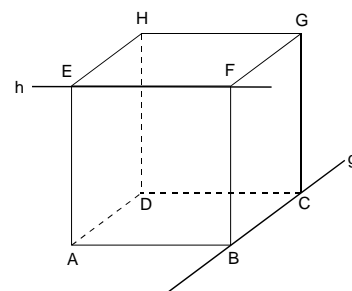
Exercise 2 (exam question)

Category C1

Given straight lines

$$g: \vec{r}_1 = \begin{pmatrix} 0 \\ -7 \\ -3 \end{pmatrix} + t \begin{pmatrix} 12 \\ 4 \\ 3 \end{pmatrix} \text{ and } h: \vec{r}_2 = \begin{pmatrix} 8 \\ 0 \\ 12 \end{pmatrix} + s \begin{pmatrix} 3 \\ -12 \\ 4 \end{pmatrix}$$

- Show that g and h are (skewed) perpendicular.
- Calculate the vertices A, B, F of a cube with edges B, C on g and E, F on h



Exercise 3**Category C1**

An unidentified flying object (UFO) passes the earth on a straight line at constant velocity. It has been observed by a radar control perpendicular above Saint Gall ($9^{\circ}22'$ east; $47^{\circ}22'$ north) at an altitude of 50'000 km.

Two hours later the UFO appears perpendicular above New York (74° west; 41° north) at a height of 30'000 km.

- What is the velocity of the UFO? (Don't consider the rotation of the earth for the moment. Earth radius 6371 km.)
- Solve a) but take into consideration the rotation of the earth.
- What is the angle between the flight direction and the tangent to the meridian direction above Saint Gall?
- When is the UFO nearest to the earth (time after passing Saint Gall)?

Exercise 4**Category C1**

An ocean liner starts in Cape Town (18.5° east; 34° south) sailing towards New York (74° west; 41° north) using the shortest route possible (earth radius 6371 km).

- How many days does the ocean liner need for the journey if its average velocity is 36 km/h?
- When does the ocean liner cross the equator?
- Why is the time needed from Cape Town to the equator, subsequently to New York, not proportional to the corresponding latitudes?
- (encroachment on other subjects) Is the shortest route from Cape Town to New York possible at all (on a globe or an atlas)? Compare the latitude and climate of Cape Town with a similar town in the northern hemisphere.

Exercise 5 (exam question)**Category C1**

A meteorite is observed perpendicularly above Zurich (24° east; 47° north) at a height of 60'000 km. Five hours later the meteorite appears at a height of 1000 km perpendicularly above the north pole (uniform continuous movement, earth radius 6371 km).

- What is the velocity of the meteorite?
- Determine the position vector, dependant on the time (Zurich: $t = 0$).
- Will the meteorite strike the earth? If the answer is yes, where will it strike (Cartesian coordinates), otherwise what is its closest distance to the earth?