

## Belgian entrance examinations for civil engineering students

There are no central examinations in secondary schools in Belgium. The following questions are examples from the entrance examinations at three Flemish Universities (Louvain, Brussels and Ghent - for the last time organized in 2003), for students who want to commence studies in civil engineering in the Faculty of Applied Sciences. In the examinations the students are allowed to use the tools they learned to work with at secondary school, including CAS. The questions are not representative for Belgium because only approximately 1000 students (mostly good in math) attempt the examination each year.

### Question 1 - Category C3

Consider the following system of linear equations in the real variables  $x, y, z$  :

$$\begin{cases} a \cdot x + y + z = 1 \\ x + a \cdot y + z = a \\ x + y + a \cdot z = a^2 \end{cases}$$

For which value(s) of the real constant  $a$  does the system have

- No solutions?
- Exactly one solution? Find this solution.
- Infinitely many solutions? Find the solutions.

#### Solution

First of all, be aware of the fact that a CAS cancels symbolic factors in the numerator and denominator, without mentioning that the symbolic factor may not be zero!

To detect the different cases, first calculate the zeros of the determinant of the matrix  $A$  of the system.

Case 1  $\det(A) \neq 0$ :  $a \neq 1$  and  $a \neq -2$

The system has exactly one solution. The solution can be found with the reduced row-echelon form of the augmented matrix of the system.

CAS interface showing the input of the system of equations:

$$\begin{cases} \frac{a}{a} & 1 \\ \frac{a^2 - b^2}{a - b} & a + b \end{cases}$$

CAS interface showing the calculation of the determinant and solving for  $a$ :

$$\det \begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{pmatrix} = (a-1) \cdot (a^2 + a - 2)$$

$$\text{solve}((a-1) \cdot (a^2 + a - 2) = 0, a)$$

$a = 1$  or  $a = -2$

$$\text{solve}((a-1) \cdot (a^2 + a - 2) = 0, a)$$

CAS interface showing the augmented matrix and its row echelon form:

$$\left[ \begin{array}{ccc|c} a & 1 & 1 & 1 \\ 1 & a & 1 & a \\ 1 & 1 & a & a^2 \end{array} \right] \rightarrow R_1$$

$$\left[ \begin{array}{ccc|c} a & 1 & 1 & 1 \\ 1 & a & 1 & a \\ 1 & 1 & a & a^2 \end{array} \right]$$

CAS interface showing the reduced row echelon form of the augmented matrix:

$$\text{rref}(M)$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & a & a^2 \\ 1 & 0 & 0 & \frac{1}{a+2} - 1 \\ 0 & 1 & 0 & \frac{1}{a+2} \\ 0 & 0 & 1 & \frac{(a+1)^2}{a+2} \end{array} \right]$$

The solution is  $x = \frac{1}{a+2} - 1$ ,  $y = \frac{1}{a+2}$ ,  $z = \frac{(a+1)^2}{a+2}$ .

**Note** The special case  $a = 1$  is not detected by row reduction as the factor  $a - 1$  is cancelled during the reduction process!

**Case 2**  $\det(A) = 0$ :  $a = 1$  or  $a = -2$

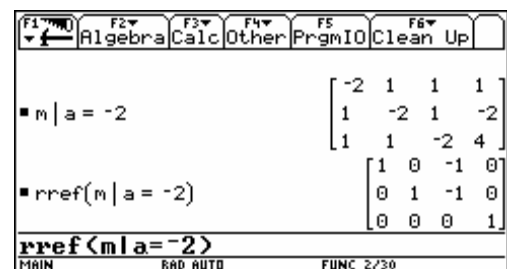
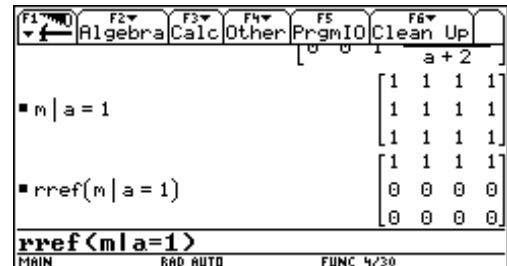
(i)  $a = 1$

The system reduces to  $x + y + z = 1$  and has infinitely many solutions:

$$\begin{cases} x = 1 - s - t \\ y = s \\ z = t \end{cases} \quad (s, t \in \mathbb{R})$$

(ii)  $a = -2$

The last equation of the reduced system is  $0 \cdot x + 0 \cdot y + 0 \cdot z = 1$ . The system has no solutions.



### Analogous exercise

Discuss the system  $\begin{cases} a \cdot x + a \cdot y + b \cdot z = 1 \\ a \cdot x + 2y + b \cdot z = 1 \\ b \cdot x + b \cdot y + a \cdot z = 1 \end{cases}$  (unknowns  $x, y, z$  and parameters  $a, b$ ).

### Question 2 - Category C0

The population migration between two geographical regions in a city, the East and the West, is as follows. Each year, 30% of the population in the East migrates to the West, while only 10% of the West moves to the East. (Neglect birth, mortality and migration to other regions).

- Give a matrix description of this situation.
- At the end of the year 2000, the East has 600000 inhabitants and the West has 400000 inhabitants. Use the matrix description to find the number of inhabitants in the East and the West in 2005 and 1999.
- If this pattern continues, find the limit situation of the population distribution in the East and the West.

### Solution

a) Let  $x_i$  be the number of inhabitants in the East and  $y_i$  the number of inhabitants in the West at the end of year  $i$ , then the number of inhabitants at the end of year  $i + 1$  becomes:

$$\begin{cases} x_{i+1} = x_i - 0.3x_i + 0.1y_i \\ y_{i+1} = y_i + 0.3x_i - 0.1y_i \end{cases} \quad \text{or} \quad \begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} = A \cdot \begin{bmatrix} x_i \\ y_i \end{bmatrix} \quad \text{with} \quad A = \begin{bmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{bmatrix} \quad (\text{the transition matrix}).$$

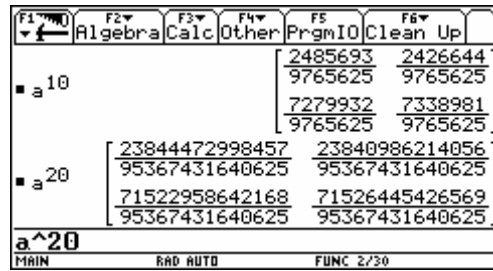
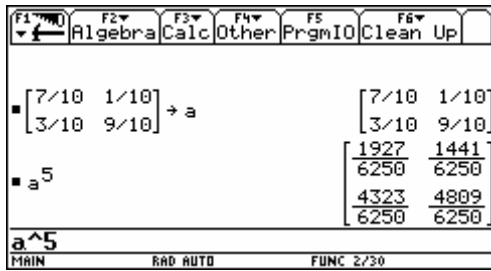
$$\text{b) } \begin{bmatrix} x_{2005} \\ y_{2005} \end{bmatrix} = A \cdot \begin{bmatrix} x_{2004} \\ y_{2004} \end{bmatrix} = A^2 \cdot \begin{bmatrix} x_{2003} \\ y_{2003} \end{bmatrix} = \dots = A^5 \begin{bmatrix} x_{2000} \\ y_{2000} \end{bmatrix}$$

$$\begin{bmatrix} x_{2005} \\ y_{2005} \end{bmatrix} = A^5 \cdot \begin{bmatrix} x_{2000} \\ y_{2000} \end{bmatrix} = \begin{bmatrix} 0.308 & 0.231 \\ 0.692 & 0.769 \end{bmatrix} \cdot \begin{bmatrix} 6000000 \\ 4000000 \end{bmatrix} = \begin{bmatrix} 277216 \\ 722784 \end{bmatrix}$$

$$\begin{bmatrix} x_{2000} \\ y_{2000} \end{bmatrix} = A \cdot \begin{bmatrix} x_{1999} \\ y_{1999} \end{bmatrix}$$

$x_{1999}$	$A^{-1}$	$x_{2000}$	1.5	-0.17	6000000	833333
$y_{1999}$		$y_{2000}$	0.5	1.17	4000000	166667

**Note:** When working in exact mode with fractions, the powers of the transition matrix  $A$  are correct but not useful:



### Extension of question 2 - Category C4

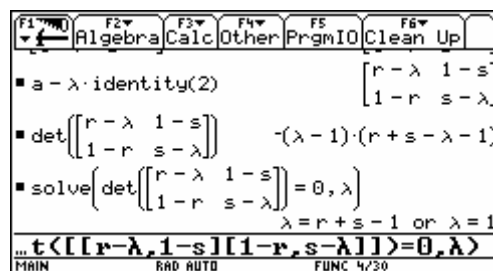
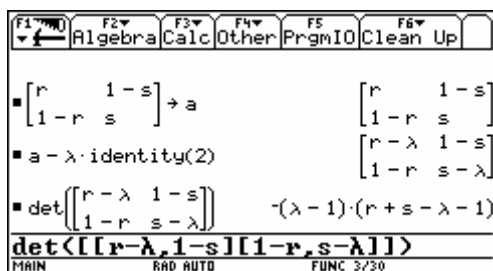
This extension was not a question on the entrance examination.

- d) Consider the transition matrix  $A = \begin{bmatrix} r & 1-s \\ 1-r & s \end{bmatrix}$  ( $0 \leq r \leq 1$  and  $0 \leq s \leq 1$ ).
- Find the eigenvalues of  $A$ .
  - Find two linearly independent eigenvectors  $X_1, X_2$  of  $A$  with corresponding eigenvalues  $\lambda_1, \lambda_2$ .
  - Form the matrix  $P$  having  $X_1, X_2$  as its column vectors and check that  $D = P^{-1}AP$  is a diagonal matrix  $A$  with  $\lambda_1, \lambda_2$  as its successive diagonal entries ( $P$  diagonalizes  $A$ ).
  - Prove that  $A^k = PD^kP^{-1}$ .
  - Find  $\lim_{k \rightarrow \infty} A^k$  and apply the result to the transition matrix of the population migration.

### Solution

- (i)  $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq 0$  is an eigenvector of  $A$  with eigenvalue  $\lambda$  if  $AX = \lambda X$  or  $(A - \lambda I)X = 0$ .

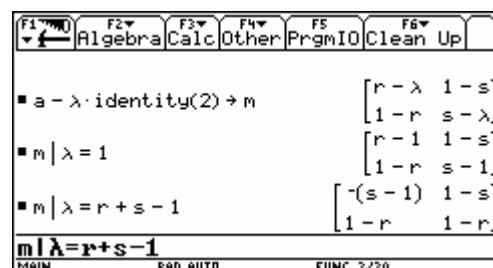
This homogeneous system of linear equations has a solution  $X \neq 0$  if and only if  $\det(A - \lambda I) = 0$ .



The eigenvalues are  $\lambda_1 = 1$  and  $\lambda_2 = r + s - 1$ .

- (ii) Case 1:  $\lambda_1 \neq \lambda_2$  or  $r + s - 1 \neq 1$   
or  $r + s \neq 2$

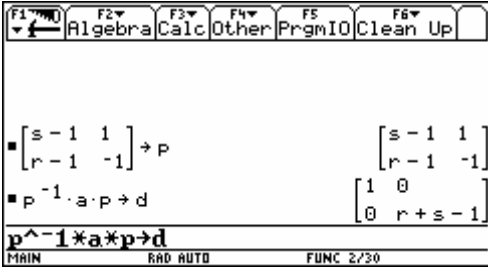
For  $\lambda_1 = 1$ , choose  $X_1 = \begin{bmatrix} s-1 \\ r-1 \end{bmatrix}$



For  $\lambda_2 = r + s - 1$ , choose  $X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .

Case 2:  $\lambda_1 = \lambda_2$  or  $r + s = 2$  or  $r = 1$  and  $s = 1$

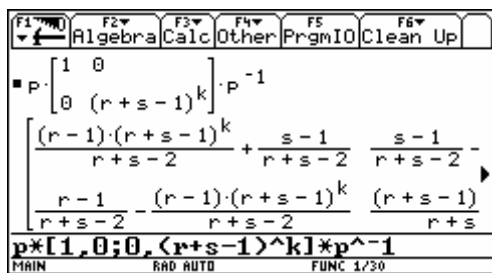
Then  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  and  $\lim_{k \rightarrow \infty} A^k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(iii) 

(iv) If  $D = P^{-1}AP$  then  $PD = AP$  and  $A = PDP^{-1}$ .

$$\begin{aligned} \text{Thus } A^k &= (PDP^{-1})(PDP^{-1})\dots(PDP^{-1}) \quad (k \text{ factors}). \\ &= PD(P^{-1}P)D(P^{-1}P)\dots(P^{-1}P)DP^{-1} \\ &= PD^kP^{-1} \end{aligned}$$

$$\begin{aligned} \text{(v) } A^k &= PD^kP^{-1} = \begin{bmatrix} s-1 & 1 \\ r-1 & -1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & (r+s-1)^k \end{bmatrix} \begin{bmatrix} s-1 & 1 \\ r-1 & -1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \frac{(r-1)(r+s-1)^k}{r+s-2} + \frac{s-1}{r+s-2} & \frac{s-1}{r+s-2} - \frac{(r+s-1)^k(s-1)}{r+s-2} \\ \frac{r-1}{r+s-2} - \frac{(r-1)(r+s-1)^k}{r+s-2} & \frac{(r+s-1)^k(s-1)}{r+s-2} + \frac{r-1}{r+s-2} \end{bmatrix} \end{aligned}$$



As  $0 \leq r \leq 1$  and  $0 \leq s \leq 1$ , it follows that  $-1 \leq r + s - 1 \leq 1$

For  $-1 < r + s - 1 < 1$ ,  $\lim_{k \rightarrow \infty} (r + s - 1)^k = 0$ ; thus  $\lim_{k \rightarrow \infty} A^k = \begin{bmatrix} \frac{s-1}{r+s-2} & \frac{s-1}{r+s-2} \\ \frac{r-1}{r+s-2} & \frac{r-1}{r+s-2} \end{bmatrix}$ .

For  $r + s - 1 = 1$  or  $r + s = 2$ ,  $\lim_{k \rightarrow \infty} A^k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  (see (ii), case 2).

For  $r+s-1=-1$  or  $r+s=0$  or  $r=0$  and  $s=0$ ,  $\lim_{k \rightarrow \infty} A^k = \lim_{k \rightarrow \infty} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^k$  doesn't exist.

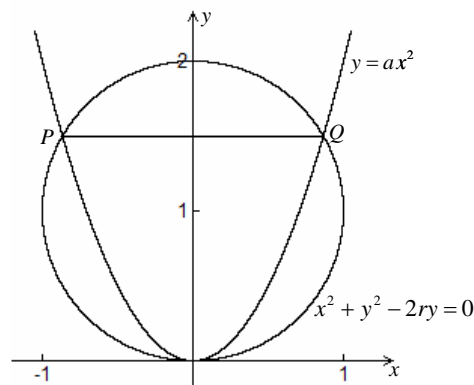
For the population migration matrix  $A = \begin{bmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{bmatrix}$  ( $r=0.7$ ,  $s=0.9$ ) the limit becomes

$$\lim_{k \rightarrow \infty} A^k = \begin{bmatrix} \frac{s-1}{r+s-2} & \frac{s-1}{r+s-2} \\ \frac{r-1}{r+s-2} & \frac{r-1}{r+s-2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{3}{4} \end{bmatrix}.$$

### Question 3 - Category C4

Consider the circle with equation  $x^2 + y^2 - 2ry = 0$  and the parabola with equation  $y = ax^2$ , where  $r$  and  $a$  are positive parameters with  $r \cdot a > 1/2$ .

- The parabola and the circle intersect at the origin and at the points  $P$ ,  $Q$ . Find the coordinates of  $P$  and  $Q$ .
- Find the area  $A$  of the region above the parabola and below the line  $PQ$ .
- Which value of  $a$  (with constant  $r$ ) gives the maximum area  $A$ ? Determine this largest area  $A_{\max}$  as a function of  $r$ .



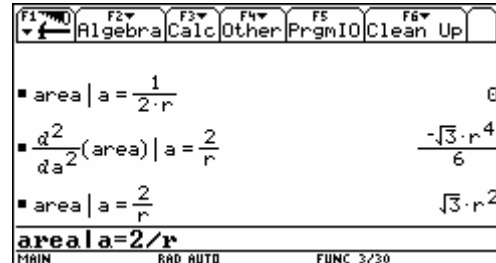
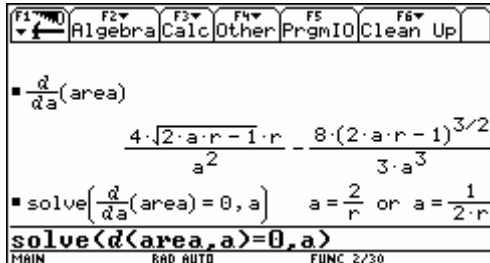
### Solution

a)  $P = \left( \frac{-\sqrt{2ar-1}}{a}, \frac{2ar-1}{a} \right)$  and  $Q = \left( \frac{\sqrt{2ar-1}}{a}, \frac{2ar-1}{a} \right)$ .

b) The area is  $A = \frac{4(2ar-1)^{3/2}}{3a^2}$ .

c) An extremum for  $A$  can occur when  $\frac{dA}{da} = 0$  or  $a = \frac{2}{r}$  (interpret the second solution  $a = \frac{1}{2r}$ ).

For geometrical reasons we expect a maximum and this is confirmed by the negative sign of the second derivative for  $a = \frac{2}{r}$ . The maximum area is  $A_{\max} = \sqrt{3}r^2$ .



Note: CAS can do the correct calculations, but the student must know that:

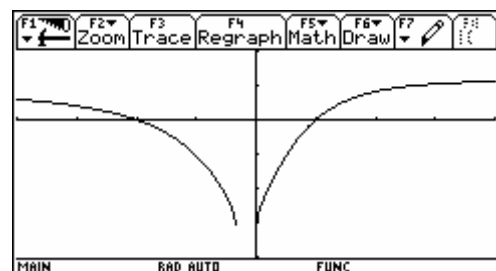
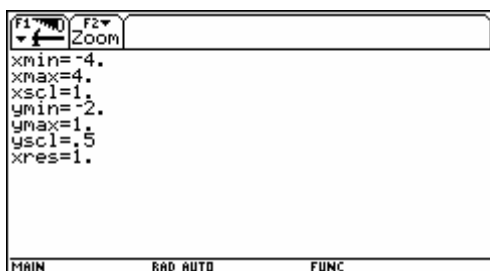
- the coordinates of P and Q can be found by solving a system of equations,
- the area of a region between two graphs can be expressed as an integral,
- an extremum of a differentiable function occurs at a point where the derivative equals zero and changes sign.

#### Question 4 - Category C0

The figure shown gives the graph of a real function  $f$ , defined as follows  $f(x) = \arcsin\left(\frac{x^2 + ax + b}{cx^2 + d}\right)$ .

with  $a, b, c, d \in \mathbb{R}$ . The graph has a horizontal asymptote with equation  $y = \frac{\pi}{6}$ , intersects the  $x$ -axis

at the points  $(1, 0)$  and  $(-2, 0)$  and intersects the  $y$ -axis at  $\left(0, -\frac{\pi}{2}\right)$ .



- Find  $a, b, c$  and  $d$ .
- Are there other asymptotes? If yes, find the equation(s).
- Determine the domain  $D$  of  $f$ .

### Solution

a) horizontal asymptote  $y = \frac{\pi}{6}$ :  $\lim_{x \rightarrow \infty} \arcsin\left(\frac{x^2 + ax + b}{cx^2 + d}\right) = \frac{\pi}{6}$  or  $\arcsin\left(\frac{1}{c}\right) = \frac{\pi}{6}$  or  $c = 2$ .

$$(1,0) \in f : \quad \arcsin\left(\frac{1+a+b}{2+d}\right) = 0 \quad \text{or} \quad 1+a+b=0$$

$$(-2,0) \in f : \quad \arcsin\left(\frac{4-2a+b}{8+d}\right) = 0 \quad \text{or} \quad 4-2a+b=0$$

$$\left(0, -\frac{\pi}{2}\right) \in f : \quad \arcsin\left(\frac{b}{d}\right) = -\frac{\pi}{2} \quad \text{or} \quad \frac{b}{d} = -1$$

It is easy to find the solution  $a = 1, b = -2, c = 2, d = 2$ . As a check, the graph can be drawn.

b) The function  $f(x) = \arcsin\left(\frac{x^2 + x - 2}{2x^2 + 2}\right)$  has no other asymptotes.

There is no oblique asymptote because the horizontal asymptote  $y = \frac{\pi}{6}$  is for  $x \rightarrow \pm\infty$ .

A vertical asymptote is impossible because the range  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$  of the arcsin-function is bounded.

c) The domain  $D$  of the function is the set of points  $x$  for which

$$\begin{aligned} -1 &\leq \frac{x^2 + x - 2}{2x^2 + 2} \leq 1 \\ &\Downarrow \\ -2x^2 - 2 &\leq x^2 + x - 2 \leq 2x^2 + 2 \quad (2x^2 + 2 \text{ is positive!}) \\ &\Downarrow \\ 3x^2 + x &\geq 0 \quad \text{and} \quad x^2 - x + 4 \geq 0 \\ &\Downarrow \\ x(3x + 1) &\geq 0 \quad \text{and} \quad x \in \mathbb{R} \\ &\Downarrow \\ x &\leq -\frac{1}{3} \quad \text{or} \quad x \geq 0 \end{aligned}$$

We conclude that  $D = \left]-\infty, -\frac{1}{3}\right] \cup [0, \infty[$ .

**Remark:** *This example illustrates that CAS is not always of practical use. CAS is only one of the many tools in mathematics and one should use the right tool at the right moment.*