Probability experiments

TARGET GROUP
Students at secondary education; high school. Students of about 15-17 years old.

TOPIC
Probability and Simulation

PRIOR MATHEMATICAL KNOWLEDGE
Rational Numbers; percentages and decimals, ratio & proportion. Simple statistics; using tables and graphs to organize and display information.

PRIOR CALCULATOR EXPERIENCE
Basic Graphing Calculator experience, having used an APP before (know how to start an APP and knowing how to use the function keys)

Reasoning about probability is difficult. Especially because certain intuitions seem to be very evident, but they can be very wrong. For example, when asking for the probability of the outcomes of tossing a coin (head or tail) – assuming the coin is fair and not tricked – most people will quickly and confidently report back with the correct response ½. But the outcome of a single random toss of the coin is unpredictable. A famous French philosopher d'Alembert said "When a coin is tossed, it has forgotten what face came up the previous time it was tossed."

This can get complicated when you do an experiment. A first toss results in tail. What will be the result for the next toss? Now the reasoning can go into two directions. Some may follow d'Alembert, but others may think that on the average the number of tails and heads will be equal, so it is more likely that the other face – head – will show up.

The same difficulty in reasoning can appear when dealing with "a large number of tosses". When a large number of tosses are done the frequency of heads and tails will be equal. But again, what is a large number?

What we can say is that in the long run the relative frequency of an event approximates its probability as close as we want but nobody can tell us after how many trials.

To reflect on this problem we will analyze later the following two situations in detail:

- CHEATING AND NOT WINNING
  The idea is to toss a coin that is modified so many times that it becomes clear that the number of times heads show up is really more than the number of tails.
- I LIKE GRAPEFRUIT!
  1000 persons have given their opinion about liking or not liking grapefruit. Of these thousand persons, 55% said they like grapefruit. When from this population of 1000, a random sample of 100 persons is selected, how many will like grapefruit? 55? Are you sure?

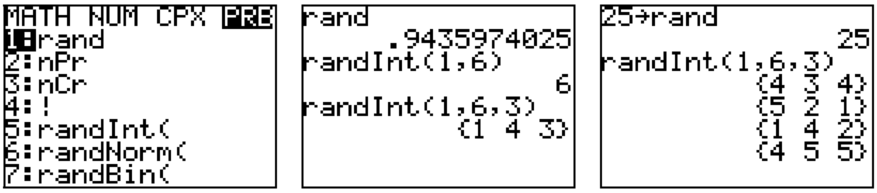
But first we will start with some fair probability experiments.

1. At random?

On the TI-84 Plus it's possible to generate at random a series of numbers with the following commands (`[MATH]<PRB>`) :

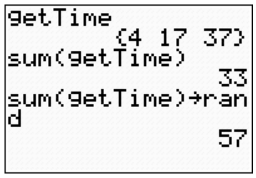
`rand`	generates a random number >0 and <1
`randInt`	generates a random integer in a specific range

Therefore the TI-84 Plus uses a procedure that starts from a seed that is stored in `rand`.



When you use the calculator for the first time or when it is reset the value of rand is 0. This can be annoying when the students do a probability experiment and all get the same results while they expect random numbers.

To avoid this let each student choose a different seed to start with, e.g. the sum of the figures of their date of birth and the date of today: 01/01/1992 gives 23 and 10/03/2006 gives 12, so together 35.

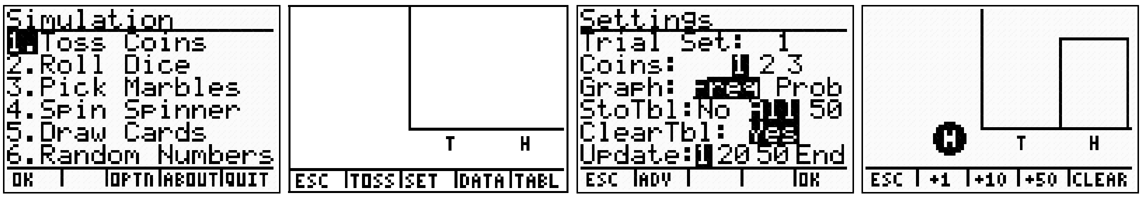


On the TI-84 Plus it's also possible to use the value of the clock as a seed to start a random procedure. Therefore you need the `getTime` command form the CATALOG.

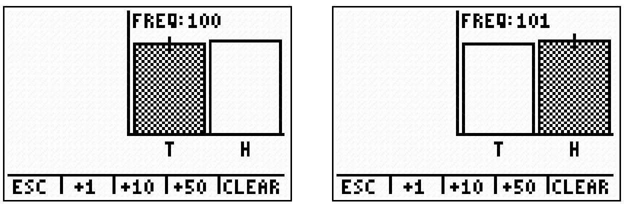
2. Coin tosses

It's very easy to simulate the toss of a coin with the application Probability Simulation and to use the results to visualize that the relative frequency of the event heads is in the long run ½, which is the theoretical probability of the event heads.

The experiment below starts with `rand` equals to 8 (`8 STO▸ rand`) [2]. Once `1.Toss Coins` is started press [`F3`], `SET`, to define the `Settings`, leave the `Settings` window, `OK`, and press [`F2`], `TOSS`, for the first toss.



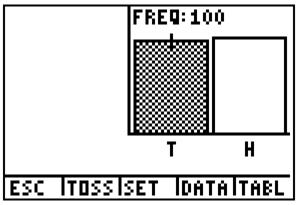
After 200 extra tosses we get the following results:

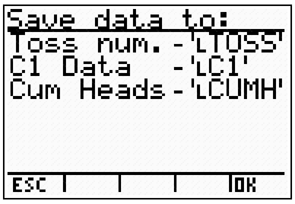


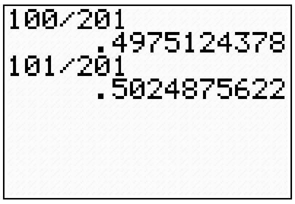
2 It is also possible to define the seed by pressing [`F3`], `OPTN`, at the menu window of Probability Simulations.

After saving the generate data [F4], DATA, some calculations show the expected result.

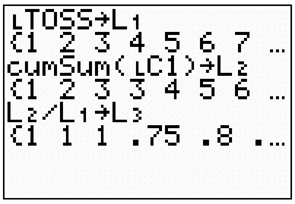
$$\text{T: } \frac{100}{201} \simeq 0{,}498 \quad \& \quad \text{H: } \frac{101}{201} \simeq 0{,}502 .$$

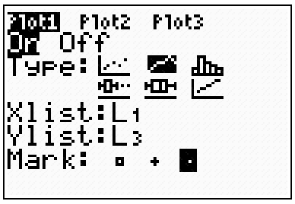


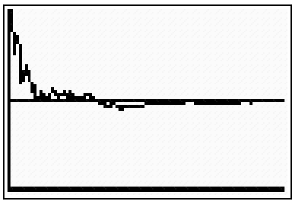




With these lists we can visualize the law of the large numbers as follows (Y1=0.5).

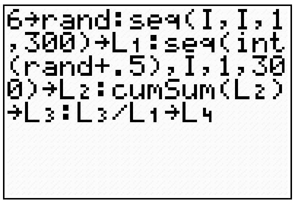


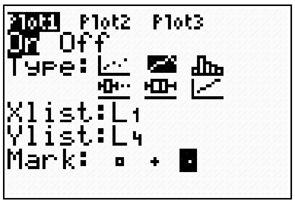


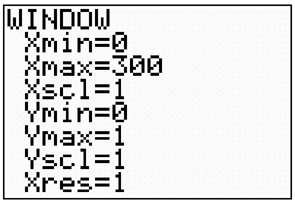


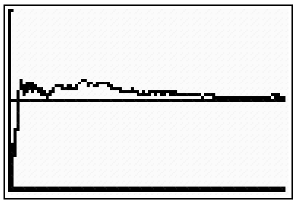
This plot shows indeed a quick “convergence” towards ½. The data fluctuate around the horizontal line $y = 0.5$.

Although Probability Simulation is a very powerful for demonstrations, it is also possible to program the example above as follows:







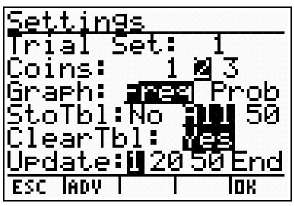


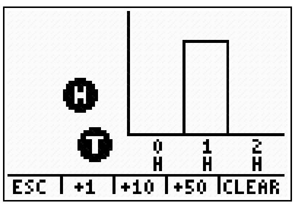
To conclude, something to think about: ‘Should we be surprised by a series of several heads or tails consecutively?’ ‘Should we therefore think that the coin is manipulated?’

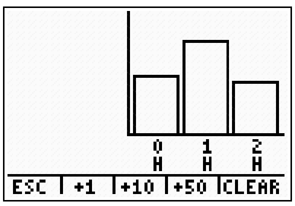
In case of tossing two coins the possible events are: all tails, one tails one heads, all heads.

Is the probability of each of these events equal to 1/3? If yes this means that in the long run the relative frequency each event will approximate 1/3.

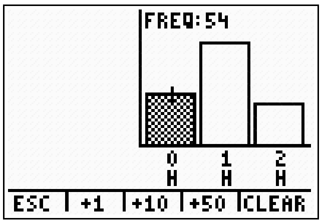
Let’s try it out with Probability Simulations (rand = 6).

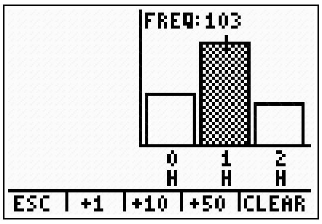


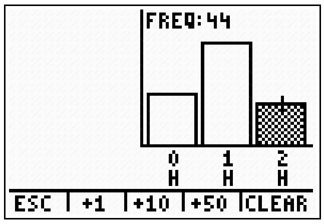




After 201 tosses we note a very clear difference between the appearance of Heads-Heads, Heads-Tails and Tails-Tails.







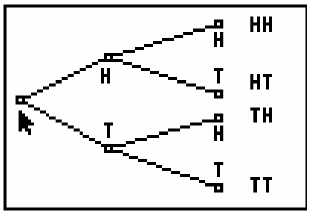
$p(\text{HH}) \approx \frac{54}{201} \approx 0.27$ $\quad p(\text{HT or TH}) \approx \frac{103}{201} \approx 0.51$ $\quad p(\text{TT}) \approx \frac{44}{201} \approx 0.22$

Remember Cardano's error during a dice game with three dice. He considered {1,3,5} and {1,5,3} as the same events to reach a sum of 9 points.

A handy tool to miss no events is a tree diagram. The tree diagram for tossing two coins tells us immediately the exact probability distribution.

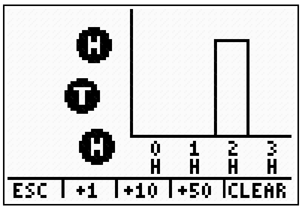
p(HH) = p(HT) = p(TH) = p(TT) = 0.25

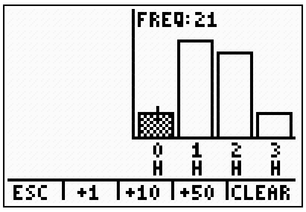
$\Rightarrow$ p{1 times Tails}=0.5



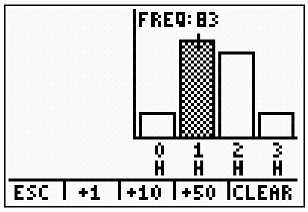
Made with Cabri Junior

And what about three coins?

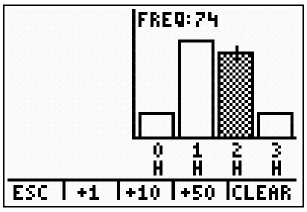




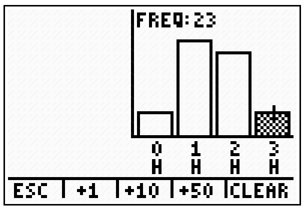
p(HHH) ≈ 0.10



p(TTH or THT or HTT) ≈ 0.37



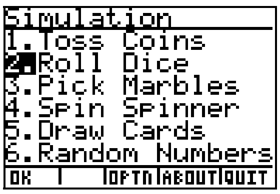
p(THH or HTH or HHT) ≈ 0.41

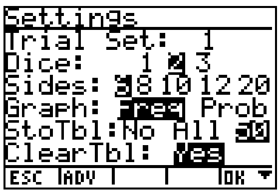


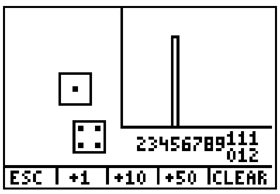
p(TTT) ≈ 0.11

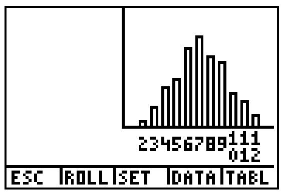
3. Throwing dice

`2.Roll Dice` you can simulate the throwing of dice, up to 3 dice. Let's start with two dice.



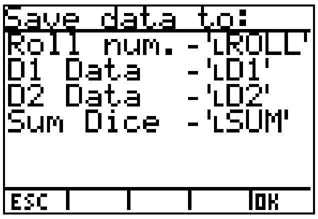


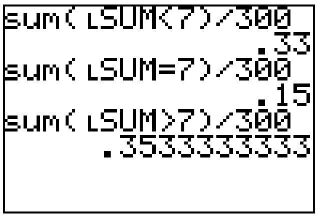




After saving the data in lists we can do several calculations.

Dice 1	Dice 2	Dice 1	Dice 2	Dice 1	Dice 2	Dice 1	Dice 2	Dice 1	Dice 2	Dice 1	Dice 2
1	1	2	1	3	1	4	1	5	1	6	1
1	2	2	2	3	2	4	2	5	2	6	2
1	3	2	3	3	3	4	3	5	3	6	3
1	4	2	4	3	4	4	4	5	4	6	4
1	5	2	5	3	5	4	5	5	5	6	5
1	6	2	6	3	6	4	6	5	6	6	6



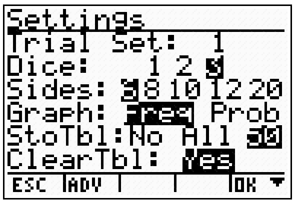


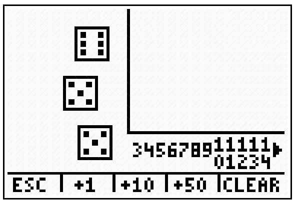
UPNOVER 7

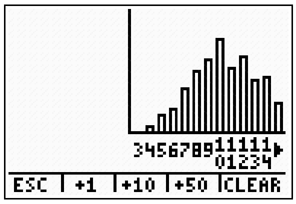
The Grand Duke of Tuscany noted during gambling that the sum of 10 points appeared more than the sum of 9 when he threw three dices. He could not explain what was happening because the amount of possibilities to get 9 is the same as to get 10.

9=1+2+6=1+3+5=1+4+4=2+2+5=2+3+4=3+3+3

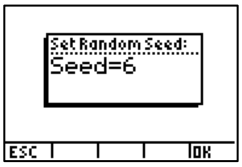
10=1+3+6=1+4+5=2+2+6=2+3+5=2+4+4=3+3+4





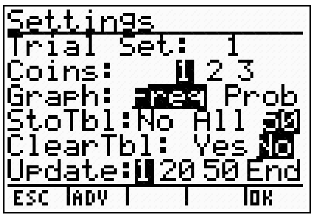


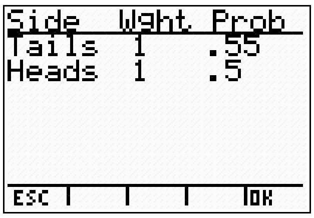
4. Cheating and not winning

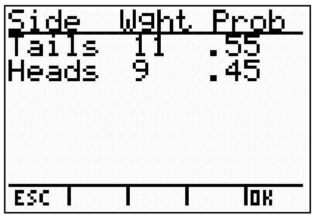


We start with initializing the rand variable. In this experiment the coins are really modified! We are going to toss a coin with the probability for tails 0.55 and for heads 0.45.

To modify the coin press [F2], ADV, in the Settings window. Don't forget to save the settings with OK, [F5].





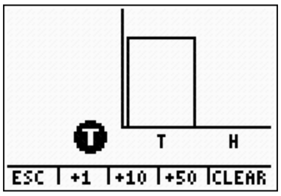


We will now gamble about the result after fifty tosses of the coin.

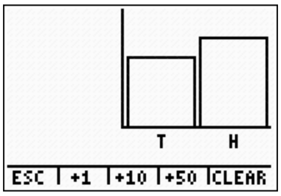
Will you choose more tails (55% chance) or more heads (45 % chance)?

This seems easy, tails of course!

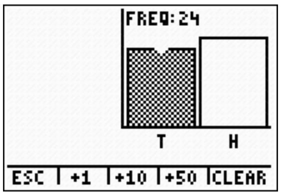
First we will toss the coin once and then we will start gambling.

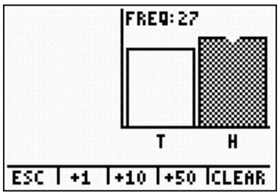


The first toss



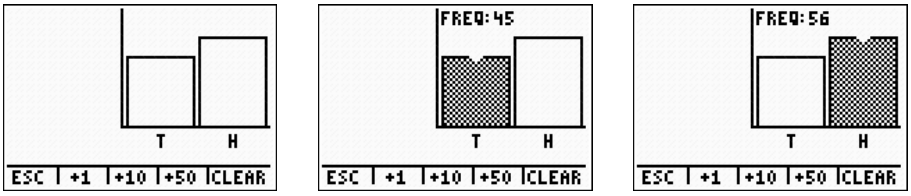
After 50 times



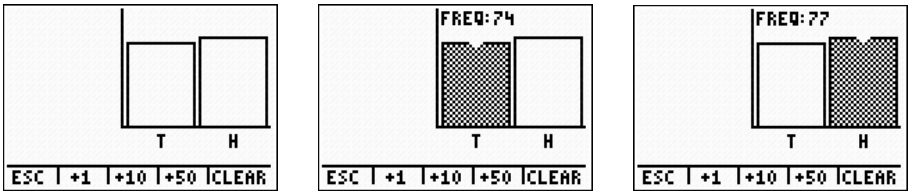


The result of the first toss seems to be OK but after 50 tosses more you lose.

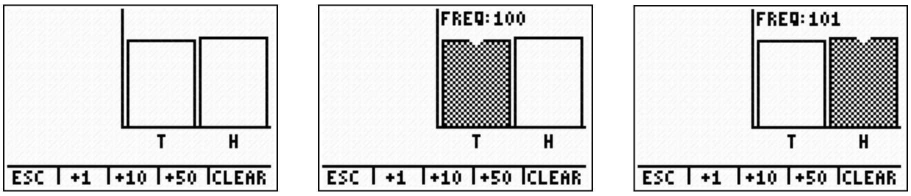
And after again 50 times more? You lose again!



Keep going on and toss the coin another 50 times. Still no win yet and for sure you cheated the coin.



Keep going on… Still lost!?



Here we stop, because the next tosses will give you the result you predicted.

The conclusion is that the results of the tossing are unpredictable, even with a cheated coin. In other words, the law of the large numbers is only true with really large numbers.

5. I like grapefruit!

For each person in a large group of 1000 persons, we know their preference: liking or not liking grapefruit. We assume the persons are honest and that when they are asked a second time will give the same answer.

A sample of 100 persons is selected form the large group. The problem we have to face is that this sample is randomly selected from the large group, so each time we select another random sample of 100 persons, the results may be different.
The question now is how to simulate this sample.

With the graphing calculator, we can design a small program that allows us to simulate the random selection of 100 persons from the large group of which 55% of the people do like grapefruit.

Below we have listed the program and the results of a series of samples. When you do the experiment yourself you will certainly get different results.

```
0→N
Input "SAMPLE ?",S
For(I,1,S)
int(rand+.55)→A
A+N→N
End
Disp N/S*100
```

```
EXEC EDIT NEW
1:FLUCTUA
```

```
prgmFLUCTUA
SAMPLE ?100
             52
           Done
SAMPLE ?100
             57
           Done
```

```
           Done
SAMPLE ?100
             52
           Done
SAMPLE ?100
             48
           Done
```

```
           Done
SAMPLE ?100
             53
           Done
SAMPLE ?100
             59
           Done
```

```
           Done
SAMPLE ?100
             60
           Done
SAMPLE ?100
             58
           Done
```

```
           Done
SAMPLE ?100
             49
           Done
SAMPLE ?100
             53
           Done
```

```
           Done
SAMPLE ?100
             59
           Done
SAMPLE ?100
             62
           Done
```

What can you say about the differences between the results and your expectation? Using experiences and insight gained from this experiment, you can reflect on surveys in general (political surveys, preferences of people, …) and the effect of repeating a survey over time.

A conclusion is that you need to take into account the fluctuation, the variation.

6. A historical note

Cardano
1501-1576

The probability theory has its roots in the world of gambling. Perhaps we can say that the first steps in the development of the probability theory were taken in search for all possible solutions of several dice games.

One of the first dissertations about probability, *Liber de Ludo Aleae*, dates from 1525 and is written by Girolamo Cardano. It is a book about throwing dice.

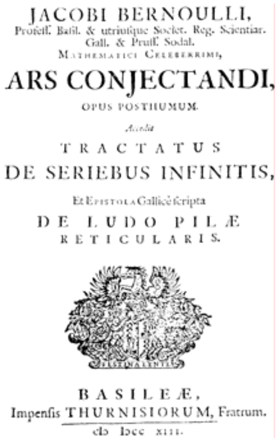
Mathematical discussions about probability arise in the seventeenth century. The correspondence between Pascal and Fermat, which consists of five letters (±1624), forms the fundaments of probability. Based on this Cristiaan Huygens wrote the first printed work on probability, *De Ratiociniis in Ludo Aleae* in 1657.

Blaise Pascal
1623-1662

Christiaan Huygens
1629-1695

The work *Ars Conjectandi* of Jacob (James) Bernoulli is a masterpiece in the history of probability. It was published posthumously by his nephew Nikolaus Bernoulli in 1713.

Jacob Bernoulli
1654-1705



JACOBI BERNOULLI,
Profess. Basil. & utriusque Societ. Reg. Scientiar. Gall. & Pruss. Sodal.
MATHEMATICI CELEBERRIMI,
ARS CONJECTANDI,
OPUS POSTHUMUM.
Accedit
TRACTATUS
DE SERIEBUS INFINITIS,
Et EPISTOLA Gallicè scripta
DE LUDO PILÆ
RETICULARIS.

BASILEÆ,
Impensis THURNISIORUM, Fratrum.
clɔ Iɔcc xiii.

It was in this masterpiece the law of the large numbers appears for the first time. James formulates and proves that the relative frequency of an event approximates its probability in the long run.

It is this idea that will be used in this article to get information from the simulations.

Another important moment in the development of probability as a modern science is the axiomatic approach of probability by Kolmogorov (1903-1987) around 1933.

Andrey Kolmogorov
1903-1987