Growth processes

TARGET GROUP
Upper secondary students.

TOPIC
Financial calculations (growth of amount of money) and exploring growth using various types of functions.

PRIOR MATHEMATICAL KNOWLEDGE
Basic knowledge of financial calculations, linear, quadratic, and exponential growth and of statistical calculations.

PRIOR CALCULATOR EXPERIENCE
Basic Graphing Calculator experience. Having used APPS before. Being familiar with Finance application, CellSheet™, statistical functions. When familiar with the application Transformation Graphing, this can be used to find the line of best fit as well.

In this section we provide five problems dealing with growth: financial calculations about growth of money deposited at the bank, financial calculations to find out the ideal age to retire, the growth of body measurements, and the variations in distance cows walk away from the stable.

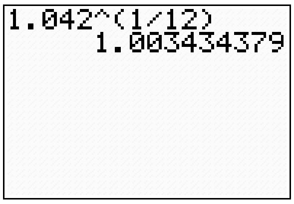
1. The joy of credit

In this example we use a spreadsheet and the geometrical progression to investigate the mathematics of credit loans.

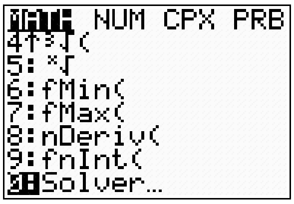
The problem
To buy a car, one borrows € 15,000 at the bank against an interest rate of 3.9% per year and an additional 0.3% for the insurance, adding up to a total of 4.2% per year. The total time period in which to pay back is set to 4 years, thus 48 months with monthly payments.

a. Calculating the monthly interest rate

The monthly interest rate equals $\sqrt[12]{1.042} = (1.042)^{\frac{1}{12}} = 1.003434...$. This can be calculated directly with the calculator or by using the Equation Solver command.



Direct calculation



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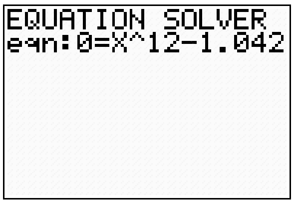
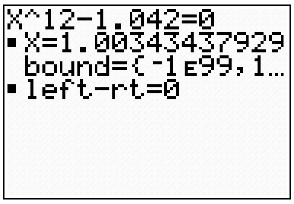


Enter the equation.



[ALPHA] [ENTER] to solve

b. Using the geometrical progression

With the geometric progression, you can calculate the value of the monthly payment as follows: $15,000 = m + m\left(1-(1+i)^{-1}\right) + m\left(1-(1+i)^{-2}\right) + ... + m\left(1-(1+i)^{-47}\right)$.

$15,000 = m\left(\frac{1-(1+i)^{-48}}{1-(1-i)}\right)$ or $m = \frac{15000 \times 0.003434}{1-(1+0.003434)^{-48}} = 339.497...$ results in € 339.50.

c. Using the spreadsheet, Cellsheet application

One can add all the effective payments in order to find an estimated value for the constant monthly payment that one has to pay.

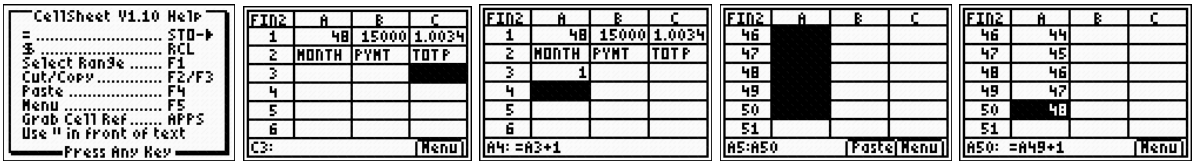
Start the application CellSheet. The start-up screen refreshes the most important features of this application.

Fill in cell A1: 48; and B1: 15000; C1: 1.0034 and leave D1 empty.

A2: MONTH; B2: PYMT; C2: TOT P; and for D2: REST.

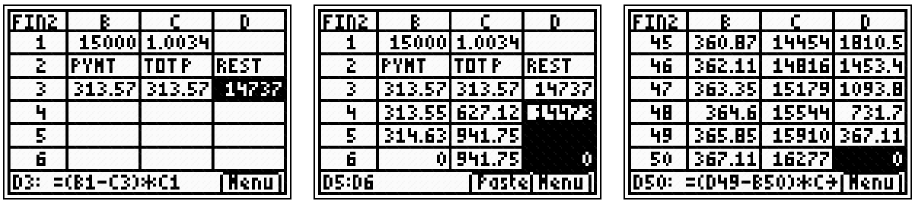
Fill in A3: 1 and for A4: =A3+1

When the cursor is in A3 press F3 (COPY) to copy the contents of A3, then press the arrow down key to get to A4. Then press F1 (RANGE) and then the down arrow all the way to A50. Next press F4 (PASTE) to paste the formula from A3 in all these cells.



Do the same for: B3: =C1*B1/A1; C3: =B3; D3: =(B1-C3)*C1 and
B4: =D3/(48-A3); C4: =B4+C3; D4: =(D3-B4)*C$1.

Copy these formulas all the way to row 50.



D50 shows there is a remaining balance of 0 to pay.

The total amount that is paid (C50) equals €16.277.

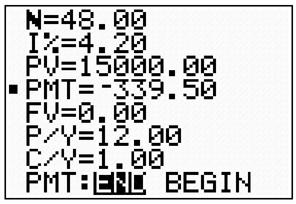
The constant monthly payment (the average of all payments) equals 16,277/48=339.10

This value is quite close to the € 339.50 that was found previously.

The rounding causes the small differences.

d. Using the Finance application

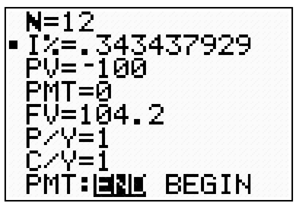
With the Finance application, one can perform the calculations quite rapidly.



Fill in the values for N, I%, PV, and P/Y.

Next place the cursor on PMT and start the solver: [ALPHA] [ENTER].

The TVM-solver returns PMT=339.50 for the monthly payment

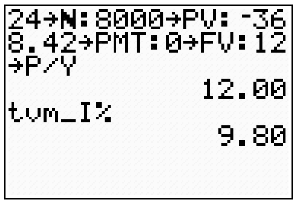


Fill in N, PV, PMT, FV and P/Y.

Next place the cursor on I% and start the solver: [ALPHA] [ENTER]. The TVM-solver returns 0.3434…%.for the interest rate per month.

Please note that when the Finance application is used, it is recommended to set the calculator in the numerical mode with 2 decimals.

e. A few more calculation



A loan of € 8000 is paid back in 2 years in monthly payments of € 368.42.

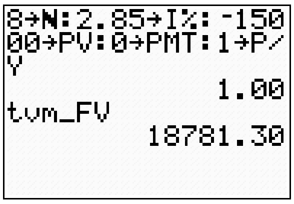
What is the yearly interest rate?

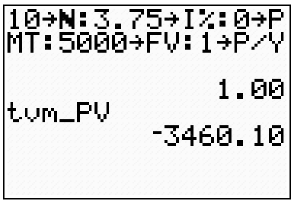
9.80 % per year is pretty high.

When an amount of €15000 is put in the bank against an interest rate of 2.85 % per year.

What will be the value of this amount after 8 years? You will gain less than €3800 in 8 years.

How much money do you need to put in the bank today, against an interest rate of 3.75% per year, to get €5000 in 10 years? The answer is €3460.10.





f. The following is problem that you may encounter yourself

A nice motorcycle is advertised for €2005. With only €205 cash at hand, this seems impossible. However the dealer is very helpful and suggests to finance the motorcycle. He has good connections with a reliable bank. The bank will give a permanent credit for an amount of €3000. This means that after registering at the bank, you can spend up to €3000. You only have to pay back a small amount per month as long as needed.

In small print is written that you pay 17.40% interest a year, or 1.45% interest per month. You decide to take a loan of €1800 from the credit account and pay back €15 per month.

The question is: How many months do you need to pay back the €1800?

There are various ways to calculate the number of months.

(i) Simple!

€1800 to be paid back in amounts of €15 per month leads to 1800/15 = 120 months, and that equals 10 years.

(ii) Forgot the interest.

The interest is 17.40%, so the total to be paid back equals 1.174 x 1800 = €2113.20. That leads to 2113.20 / 15 = 140.88 months, equals 11.74 years, and that is about 11 years and 9 months.

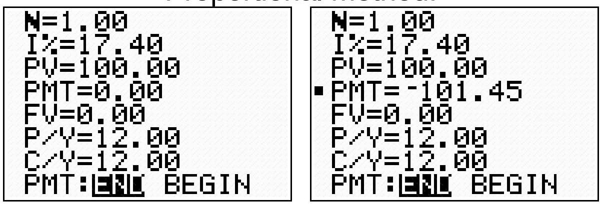
(iii) But interest is to be paid over the remaining credit amount.

This gets complicated. Each year an amount of 12 times €15 is paid back. So, after the first year left to be paid back is 1800 – 180 = €1620. So for the second year 17.4% interest is to be paid over €1620. That equals €281.88. That is more than what was paid back the first year! How is that possible?

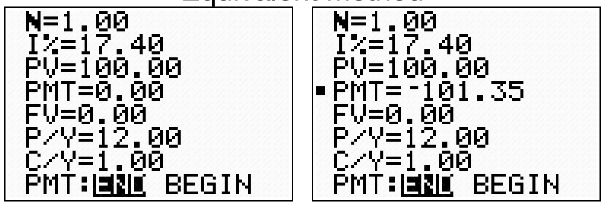
Which of the three calculations is best for this situation?

Before we elaborate on the third situation, let's investigate if 17.40% interest a year is the same as 1.45% interest per month for 12 months.

Proportional method:



Equivalent method



Let's start with a loan of €100 and a period of paying back of 1 year. The yearly interest is 17.40%. Adjusting the numbers for the payment, and payment/year and the compounding periods per year, you will notice that there is a difference when the compounding periods per year are set to 1 or 12.

Which one is correct 1.45% per month or 1.35% per month?

<u>An explanantion</u>

12 times 1.45 equals 17.40%, but actually you pay 1.45% per month over the remaining credit amount. So, that is over a year $(1.45)^{12}$ = 18.86%.

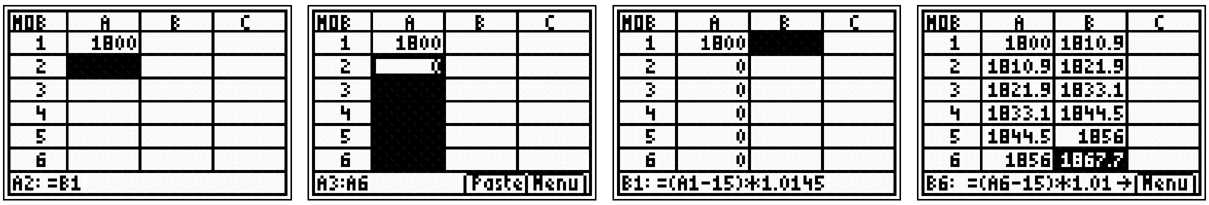
So, actually you should pay per month 1.35% so over a year you pay 17.40% because $(1.35)^{12}$ = 17.40.

Now we go back to the problem of the motorcycle. With CellSheet, we can create a table to see what happens with the amount of €1800 (bank) when we pay back €15.00 each month and use a monthly interest of 1.45%.

In column A we list the amount that is on the bank at the beginning of each month. And column B contains the amounts at the end of each month.

(i) `A1=1800` and `A2=B1`.
Copy `A2` from `A3` to `A6`.

(ii) `B1=(A1-15)*1.0145`
Copy `B1` from `B2` to `B6`



Use Finance or Cellsheet to answer the following questions:

- What is the minimal amount to pay back to the bank each month?
- How much time (number of months) is needed to pay back the complete loan of €1800 when the monthly payment is €75.00?

2. Financial calculations, what age to retire?

We start with the calculation of constant annuities at different expiry terms using the geometrical progression, and a fixed interest per period.

Let $a_1 = a$ and $a_2 = aq$ (inflation) where $q = 1+i$ (i being the interest) and n the number of periods.

Period	0	1	2	3	…	$n-1$	n
Value for period	0	a_1	a_2	a_3	…	a_{n-1}	a_n
Value for period	0	a	aq	aq^2	…	aq^{n-2}	aq^{n-1}

The total sum (sum of the terms of a geometrical progression, where $q \neq 1$ … otherwise the progression would not be geometrical!) is:

$$V_n = a + a.q + a.q^2 + ... + a.q^{n-1} = a(1 + q + q^2 + ... + q^{n-1}) = a\frac{1-q^n}{1-q}$$

$$V_n = a\frac{1-(1+i)^n}{1-(1+i)} = a\frac{1-(1+i)^n}{-i} = a\frac{(1+i)^n - 1}{i}$$

Since $V_n = V_0(1+i)^n$ substitution in the previous formula gives:

$$V_0 = a\frac{(1+i)^n - 1}{i \times (1+i)^n} = a\frac{1-(1+i)^{-n}}{i}$$

Therefore:

- $V_0 = a \times \frac{1-(1+i)^{-n}}{i}$ where V_0 calculated as a function of a **future value** for which depreciation of money is taken into account (decline in the value of money).
- $V_n = a \times \frac{(1+i)^n - 1}{i}$ where V_n is the **present value** of the capital (or the total final value).

This relationship is translated on the calculator as follows: FV for V_n, PV for V_0, PMT for *an* annuity (or *m* monthly payment), I% for $100 \times i$ and N for *n* the number of payment periods.

a. The problem situation

In a given country, the (theoretical!) age of retirement is 60 years old. There is a certain disincentive against actually taking retirement at 60, because the sum paid to the retiree is only complete (100%) if (s)he enters retirement at 65. If a person retires from work at 64 years old, the amount paid is cut by 4% (also for the payments after 65 years). Similarly, for retirement at 63 the reduction is 8%, and 12% for retirement at 62, 16% at 61 and 20% for retirement at the legal age of 60…

We assume inflation at 2.5% per year (decline in the value of money) and life expectancy of 20 years after the age of 60.

We also assume that the payments made annually by the pension fund at the due terms are appreciated (i.e. increased in value, indexed to increases in wages) by 1.5 % (i.e. less than inflation).

b. Understanding the problem

- We are comparing the total amount paid by the pension fund to the same person retiring at 60, 61, 62, 63, 64 or 65 years old.
- We do not take into account the amount received by the person continuing to work beyond the age of 60 up to the date of retirement.
- We calculate the “final” value, i.e. at age 80, of the total sum received. It is therefore the calculation of an annuity as a function of a future value.

The first annual payment is therefore made at the end of the 60^{th} year for a person retiring on reaching 60 years old, and the last payment is made just before the person’s 80^{th} birthday (life expectancy…). This person may also of course continue to live and receive his/her pension after 80, remaining blithely ignorant of statistical averages …

c. Calculations by hand

EXCLUDING APPRECIATION the following formula would be directly applicable (calculation of V_0 because future value):

$$V_{60}=V_{60}=(1-0.20)\times a\times\frac{1-(1.025)^{-20}}{0.025}=0,80\times a\times\frac{1-(1.025)^{-20}}{0.025}\approx 12.47a\ .$$

Similarly, for the other years, (taking into account annual inflationary distortion):

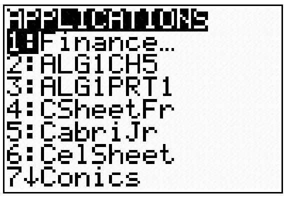
$$V_{61}=0.84\times a\times\frac{1-(1.025)^{-19}}{0.025}\times(1.025)^{-1}\approx 12.28a \qquad V_{62}=0.88\times a\times\frac{1-(1.025)^{-18}}{0.025}\times(1.025)^{-2}\approx 12.02a$$

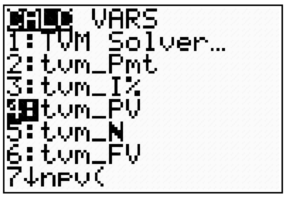
$$V_{63}=.92\times a\times\frac{1-(1.025)^{-17}}{0.025}\times(1.025)^{-3}\approx 11.71a \qquad V_{64}=0.96\times a\times\frac{1-(1.025)^{-16}}{0.025}\times(1.025)^{-4}\approx 11.35a$$

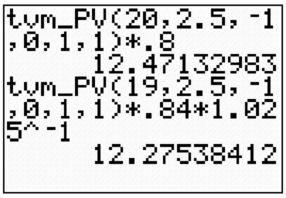
$$V_{65}=a\times\frac{1-(1.025)^{-15}}{0.025}\times(1.025)^{-5}\approx 10.94a$$

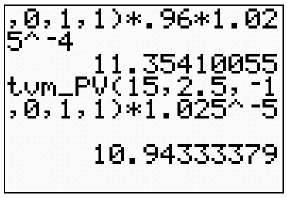
Since the values are decreasing, it would be better to retire at 60.

Use of the Finance application:









The same values are obtained, subject to rounding. The conclusion is identical.

INCLUDING APPRECIATION (for information purposes, to understand financial calculations), we get the following situation:

Let $a_1=a$ and $a_2=aqp$ (inflation) where $q=1+i$ (i being the interest), n the number of periods and $p=1+k$ (appreciation).

Period	0	1	2	3	…	$n-1$	n
Value for period	0	a_1	a_2	a_3	…	a_{n-1}	a_n
Value for period	0	ap^{n-1}	aqp^{n-2}	aq^2p^{n-3}	…	$aq^{n-2}p$	aq^{n-1}

V_n is a geometrical progression with a ratio of $\frac{q}{p}$, and scale factor p^{n-1}. Therefore:

$$V_n=a\frac{1-\frac{q^n}{p^n}}{1-\frac{q}{p}}=a\frac{\frac{p^n-q^n}{p^n}}{\frac{p-q}{p}}=a\frac{p^n-q^n}{p^n}\times\frac{p}{p-q}=ap^{n-1}\times\frac{p^n-q^n}{p-q}=ap^{n-1}\times\frac{q^n-p^n}{q-p}$$

Since $V_n=V_0(1+i)^n$ we have $V_0=a\times(1+i)^{-n}\times\frac{(1+i)^n-(1+k)^n}{i-k}$.

We now simply have to apply this formula, taking into account changes in the number of years:

$$V_{60} = 0.80 \times a \times 1.025^{-20} \times \frac{(1.025)^{20} - (1.015)^{20}}{1.025 - 1.015} \approx 14.24a$$

$$V_{61} = 0.84 \times a \times 1.025^{-20} \times \frac{(1.025)^{19} - (1.015)^{19}}{0.01} \approx 13.26a$$

$$\vdots$$

$$V_{65} = a \times 1.025^{-20} \times \frac{(1.025)^{15} - (1.015)^{15}}{0.01} \approx 9.67a\,.$$

The values start to decrease, so the best choice is to stop working and retire at the age of 60 years.

It is not possible to use the Finance application in this situation. Here, two interest rates are mixed and going in opposite directions. The Finance application uses only one rate, which is in general sufficient.

3. Body measurements

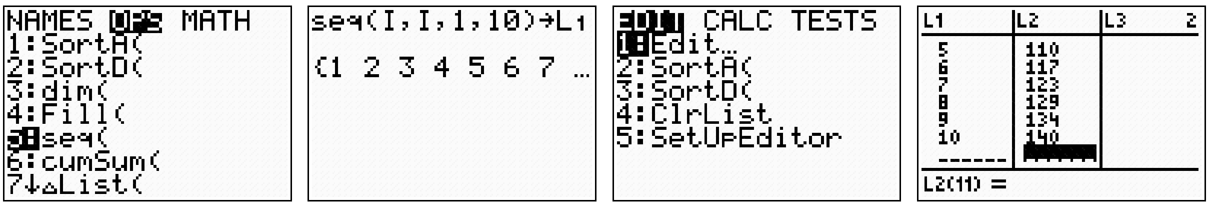
Aurelia's father – a mathematics teacher – measured various body measurements since his daughter's birth. On her tenth birthday, Aurelia asked her father if it would be possible to predict the size of her waist for when she will be 11 years old and on her twelfth birthday.

The measurements of previous years are in the following table:

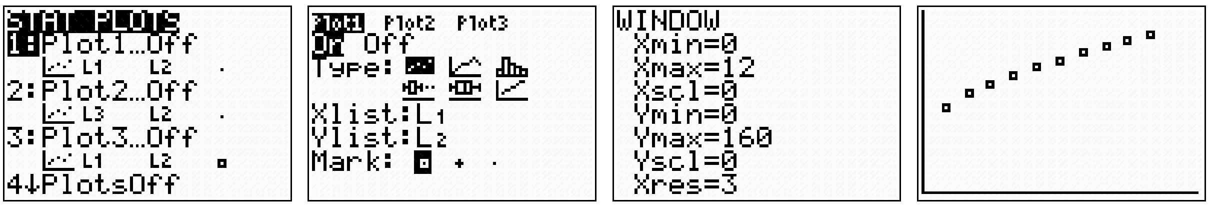
years	1	2	3	4	5	6	7	8	9	10
waist (cm)	76	87	96	104	110	117	123	129	134	140

With the TI-84 Plus you can plot scatter plots and find the line that best fits the series of points. Deciding on the visual representation which line is best is quite subjective. Therefore, we will use the correlation coefficient as a measurement of fit.

We will enter the data as follows in the lists L1 and L2:

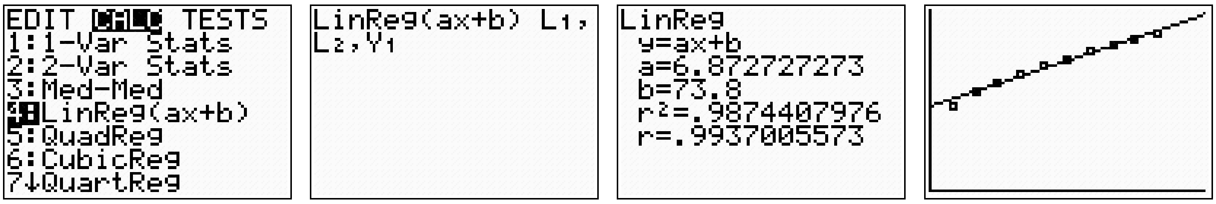


And we become the following scatter plot:



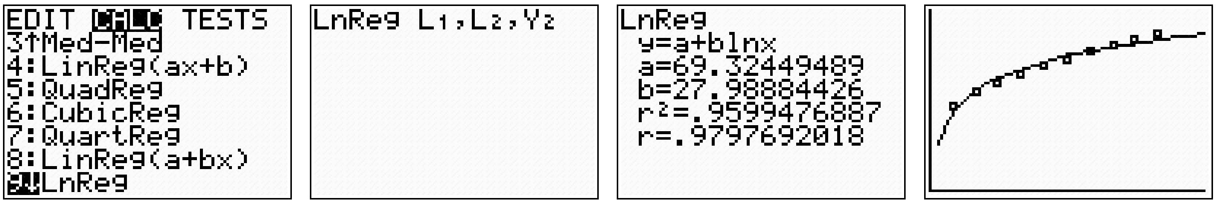
Now various kinds of formulas (functions) can be explored to see which one fits best the data. We will use the following kinds of regression: linear, logarithmic function, exponential and finally cubic regression.

Linear regression

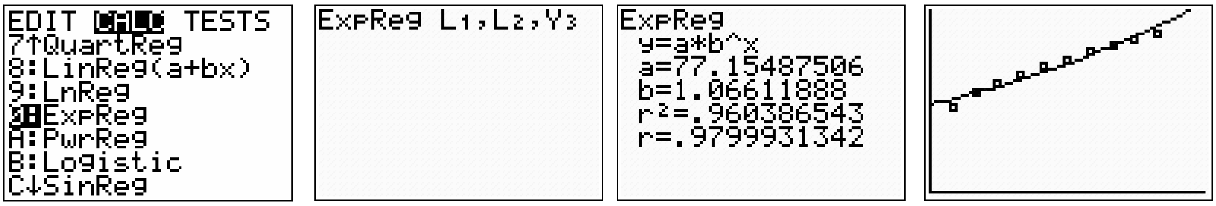


When the correlation coefficient is not displayed, you need run the `DiagnosticOn` command from the `CATALOG`.

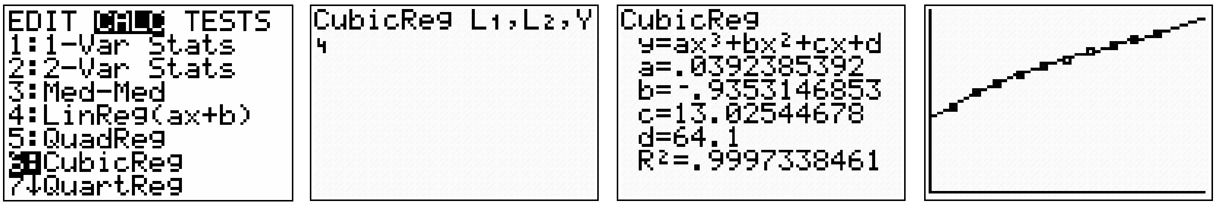
Logarithmic regression



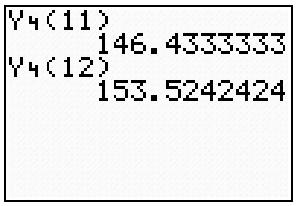
Exponential regression



Cubic Regression



This cubic model is quite good. Therefore we use this model to answer our question and to predict Aurelia's size of her waist real when she will be 11 and 12 years old.



4. Moving cattle and logistics

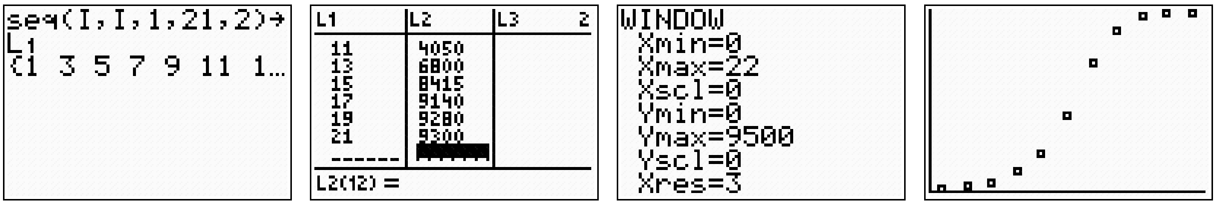
In the winter, cattle lives on the plains in the valleys and in the winter, cattle moves up in the mountains. Every morning the cows leave the stable and each night they return to be milked and to get some concentrates (extra food). Four cows from the herd, the leaders of the pack, have gotten a little clock and a GPS to determine their position. The other cows are used to follow (one of) the four leaders.

The average maximal distance between the stable and the four groups of cows is calculated each day. The table below shows these distances for every other day. From the table, you can state that after a certain number of days the maximum distance stays the same. Probably this is the distance that the cows can walk to be back in time for the milking and treats (the concentrates).

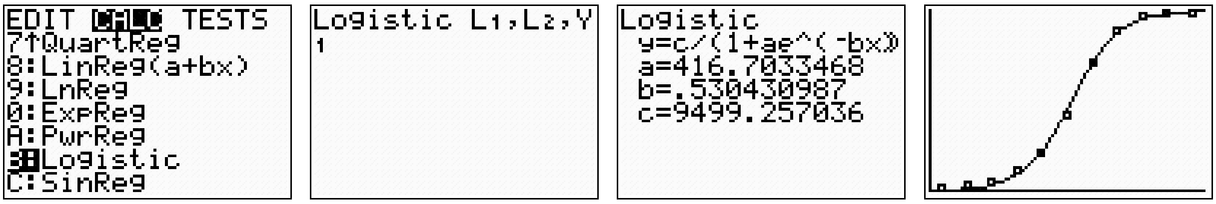
The task is to use the statistical application to investigate these data and find a curve that best fits the scatter plot that graphically represents the data.

Days	1	3	5	7	9	11	13	15	17	19	21
Distance (m)	140	270	520	1120	2015	4050	6800	8415	9140	9280	9300

We will enter the data in the lists L1 and L2 and make a scatter plot.



The scatter plot of the data consist of two parts: the first half shows an increasing growth and the second part a process of decreasing growth that almost stops growing. For such a process, the logistic function can be used to describe the growth.



Phenomena of increasing growth, followed by decreasing growth (like chemical processes, in business, animals of plants, …) can often be described with a logistic function.