Functions from real experiments

TARGET GROUP
Students at upper secondary education, high school students of about 15 - 17 years old.

TOPIC
Physics, limited growth, mechanics, energy conservation, different real functions

PRIOR MATHEMATICAL KNOWLEDGE
Linear, quadratic, broken, power and exponential functions, growth and decay processes, use of tables and graphs to organize and display information, some experience with geometric transformations, sine functions

PRIOR CALCULATOR EXPERIENCE
Basic Graphing Calculator experience, know how to start an APP and know how to use the function keys

---

The Vernier EasyData^{TM} application is a data collection application for the TI-83 Plus and TI-84 Plus (Silver Edition). EasyData supports a series of sensors and data collection systems (like CBR 2™, CBL 2™, Vernier's EasyTemp™, …) and is easy to use.

If you use EasyData with calculators from the TI-83 Plus family for data collection, use the I/O port of the calculator. You can connect the CBR or CBR 2 directly; for other sensors you need a CBL 2.

The USB-port of calculators from the TI-84 plus family provides an additional option for data collection: You can connect EasyTemp temperature sensor directly to the USB port of the calculator which simplifies data collection substantially. For other sensor you need the EasyLink™ as an adaptor. When the EasyData application is installed, the application starts automatically and you collect data and analyze and process the data with your calculator. `

The following five examples were used in the classroom and illustrate the didactical potential of the combination of simple data collection and advanced data processing with the help of the calculator. We will show how:

- concepts of physics are tested by visualization and interpretation of data and how
- mathematical models are developed to describe physical experiments.

The user learns to use EasyData from setting up and doing the experiment, through modeling, analyzing and visualizing the data. The example illustrates how the classroom situation can be modified with these new technologies. This technology can also be used for measurements outside the classroom; it is light weight, easy to take with you and can be used everywhere.

Students have a stronger relationship with data measured by themselves than with data presented in a textbook. This may lead to the situation, where all students want to participate in data collection and thus to improved students' participation. This might be challenging for the teacher who has typically only one set of instruments at his disposal. As data can be transferred between calculators, it is feasible to do parallel analysis or as another option the teacher can analyze the data together with the class by using the view-screen.

Compared to the traditional instruments used in the classroom, e.g. thermometer or stop watches, more data can be more precisely collected and the shape of the corresponding curves is obtained easier and faster. Thus, students need less time for data collection and have more time for analysis, investigation and interpretation of data.

Students can investigate variation and the effect of repeated measurements in the so-called what-would-be-if scenarios, which is an additional benefit. Students can analyze the data both algebraically and graphically and associate these relationships with mathematical functions. Finally they can use the data to find the best fit functions and discover the physical meaning of different coefficients and parameters.

1. The bouncing ball

a. Introduction

The height of a bouncing ball is continuously measured with a distance measuring device (CBR 2) connected to the calculator (TI-84 Plus) and the data collected will be analyzed. The measured movement of the ball is described as a function of time and the gravity law is derived. With energy calculations, insights can be gained where energy is lost during bouncing.

In the classroom the following questions can be asked:

- What is the highest speed of the ball and when does it occur?
- What is the acceleration during falling?
- Which function describes the distance (height) of the ball?
- Is there a model to describe the height of the ball as a function of time?
- How can the total distance of the ball be determined?
- What processes determine the “bouncing back” of the ball from the floor?
- How does the rebound height decrease from one bounce to the next?
- Can you determine how high a ball will rebound on each bounce and make predictions about its motion?

b. Didactic concepts and methodological hints

The ball is a freely falling and bouncing object where air friction is neglected. Therefore only gravity affects the ball’s movements which show that acceleration is approximately constant. The time-distance graphs are parabolic functions, which can be described by the quadratic equation $y = a(x-b)^2 + c$ where the highest point is described by the coordinates (b, c) with c as the maximum height and b as the corresponding time. The parameter a represents mathematically the shape of the parabola and depends physically on the degree of acceleration caused by gravity, which is constant during the experiment.

The curves obtained for the time-distance graphs of the individual bounces are first adjusted manually – by determining the parameters b and c and by varying parameter a.

After selecting an individual bounce with `2nd[LIST]<OPS> 8:Select(` and quadratic regression, the function describing the ball’s movement is obtained with the help of regression analysis. With the help of the application Transformation Graphing you can do the curve fitting process too.

The maximum height decreases exponentially from bounce to bounce for each ball and its initial height. For $y = hp^x$, y is the current height, h is the initial height, p is a constant depending on the properties of the ball and the floor and x is the number of the bounce.

For $x = 0$, $y = h$ (the initial height of the ball, from which it has been dropped). The coefficients of the equation describing the exponential function are determined from the data collected. The experiment can be repeated with different balls, heights and floor types.

In the time-velocity diagram the total distance for a certain time interval is represented by the area under the graph.

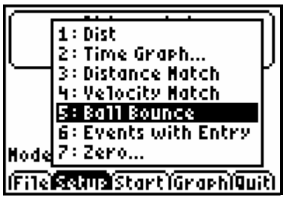
c. Performing the experiment

For the experiment, an inflatable ball with about a 25 cm diameter and about an 80 g mass is used.

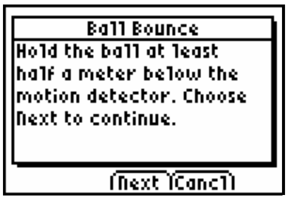
The CBR 2 is held at about 160 cm above the floor and the ball is positioned roughly 50 cm below the CBR 2 and then dropped.

Students create height-time-plots for a bouncing ball and explain how the ball's height changes mathematically from one bounce to the next.

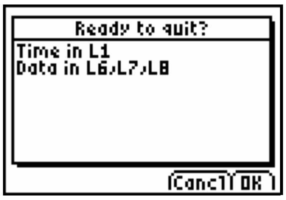
Data collection starts through the main screen of the application EasyData with the CBR 2 using the option `Ball Bounce`. The option `Ball Bounce` from the EasyData APP transforms the data into floor-bounce distances. The data are stored in lists `L1` and `L6` to `L8`, which are shown on the screen after quitting EasyData.



```
     1: Dist
     2: Time Graph...
     3: Distance Match
     4: Velocity Match
     5: Ball Bounce
     6: Events with Entry
Mode 7: Zero...
File Setup Start Graph Quit
```

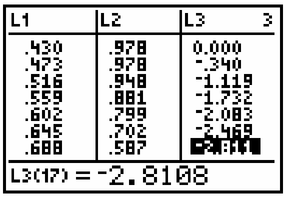


```
        Ball Bounce
Hold the ball at least
half a meter below the
motion detector. Choose
Next to continue.
                Next  Cancl
```

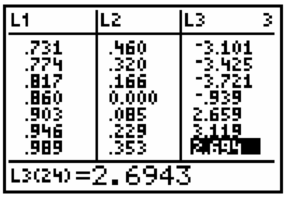


```
        Ready to quit?
Time in L1
Data in L6,L7,L8
                Cancl  OK
```

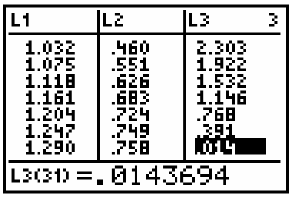
The following are data from a real experiment. We use these data to illustrate how Transforming Graphing can be used. In the example the ball was dropped at 0.473 s and we only used data up to four seconds. We have placed the distance (`L6`) in list `L2` and the velocity (`L7`) in `L3`.



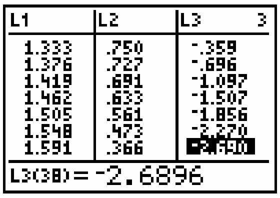
```
L1      L2      L3      3
 .430    .978    0.000
 .473    .978    -.340
 .516    .948    -1.119
 .559    .881    -1.732
 .602    .799    -2.083
 .645    .702    -2.469
 .688    .587    -2.811
L3(17) = -2.8108
```



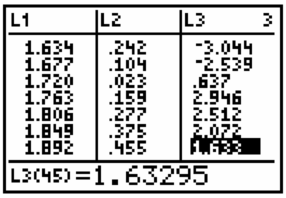
```
L1      L2      L3      3
 .731    .460    -3.101
 .774    .320    -3.425
 .817    .166    -3.721
 .860    0.000   -.939
 .903    .085    2.659
 .946    .229    3.119
 .989    .353    2.694
L3(24) =2.6943
```



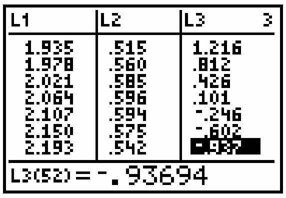
```
L1      L2      L3      3
 1.032   .460    2.303
 1.075   .551    1.922
 1.118   .626    1.532
 1.161   .683    1.146
 1.204   .724    .768
 1.247   .749    .391
 1.290   .758    .014
L3(31) =.0143694
```



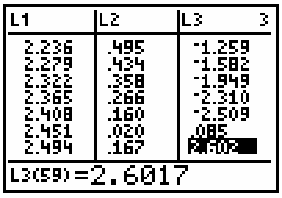
```
L1      L2      L3      3
 1.333   .750    -.359
 1.376   .727    -.696
 1.419   .691    -1.097
 1.462   .633    -1.507
 1.505   .561    -1.856
 1.548   .473    -2.270
 1.591   .366    -2.690
L3(38) = -2.6896
```



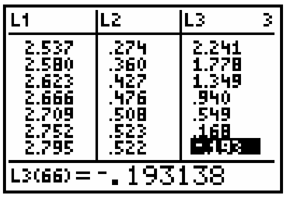
```
L1      L2      L3      3
 1.634   .242    -3.044
 1.677   .104    -2.539
 1.720   .023    .637
 1.763   .159    2.946
 1.806   .277    2.512
 1.849   .375    2.072
 1.892   .455    1.633
L3(45) =1.63295
```



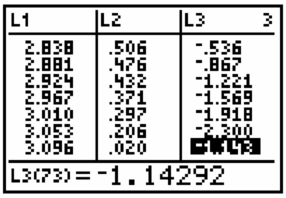
```
L1      L2      L3      3
 1.935   .515    1.216
 1.978   .560    .812
 2.021   .585    .426
 2.064   .596    .101
 2.107   .594    -.246
 2.150   .575    -.602
 2.193   .542    -.937
L3(52) = -.93694
```



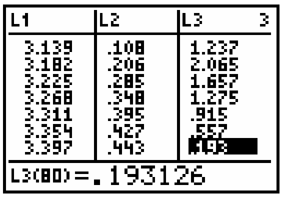
```
L1      L2      L3      3
 2.236   .495    -1.259
 2.279   .434    -1.582
 2.322   .358    -1.949
 2.365   .266    -2.310
 2.408   .160    -2.509
 2.451   .020    .085
 2.494   .167    2.602
L3(59) =2.6017
```



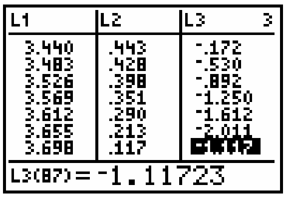
```
L1      L2      L3      3
 2.537   .274    2.241
 2.580   .360    1.778
 2.623   .427    1.349
 2.666   .476    .940
 2.709   .508    .549
 2.752   .523    .168
 2.795   .522    -.193
L3(66) = -.193138
```



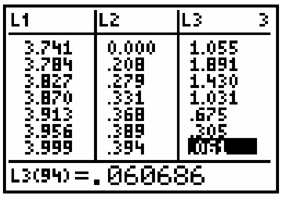
```
L1      L2      L3      3
 2.838   .506    -.536
 2.881   .476    -.867
 2.924   .432    -1.221
 2.967   .371    -1.569
 3.010   .297    -1.918
 3.053   .206    -2.300
 3.096   .020    -1.143
L3(73) = -1.14292
```



```
L1      L2      L3      3
 3.139   .108    1.237
 3.182   .206    2.065
 3.225   .285    1.657
 3.268   .348    1.275
 3.311   .395    .915
 3.354   .427    .557
 3.397   .443    .193
L3(80) =.193126
```



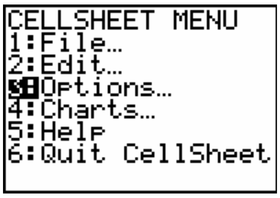
```
L1      L2      L3      3
 3.440   .443    -.172
 3.483   .428    -.530
 3.526   .398    -.892
 3.569   .351    -1.250
 3.612   .290    -1.612
 3.655   .213    -2.011
 3.698   .117    -1.117
L3(87) = -1.11723
```



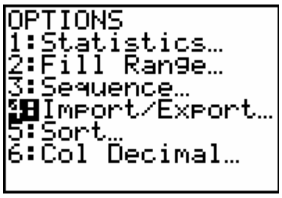
```
L1      L2      L3      3
 3.741   0.000   1.055
 3.784   .208    1.891
 3.827   .279    1.430
 3.870   .331    1.031
 3.913   .368    .675
 3.956   .389    .305
 3.999   .394    .061
L3(94) =.060686
```

With CellSheet™ we can also manipulate the data. In the following example we imported the Lists `L1` and `L2` in CellSheet columns `A` and `B`.

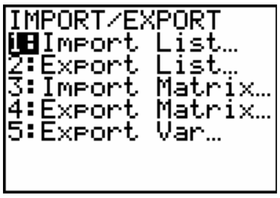
The CBR 2 was at a height of 1.60 m and in column `C` we computed the distance of the ball to the CBR 2 sensor.



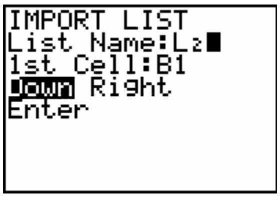
```
CELLSHEET MENU
1:File…
2:Edit…
3:Options…
4:Charts…
5:Help
6:Quit CellSheet
```



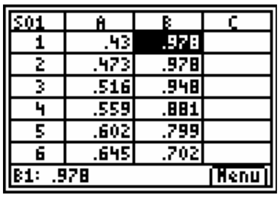
```
OPTIONS
1:Statistics…
2:Fill Range…
3:Sequence…
4:Import/Export…
5:Sort…
6:Col Decimal…
```



```
IMPORT/EXPORT
1:Import List…
2:Export List…
3:Import Matrix…
4:Export Matrix…
5:Export Var…
```

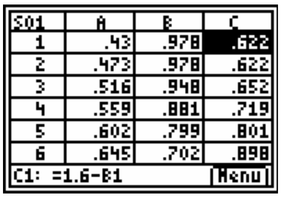


```
IMPORT LIST
List Name:L2
1st Cell:B1
Down Right
Enter
```



S01	A	B	C
1	.43	.978	
2	.473	.978	
3	.516	.948	
4	.559	.881	
5	.602	.799	
6	.645	.702	

B1: .978 Menu



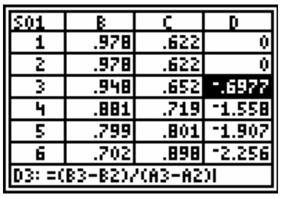
S01	A	B	C
1	.43	.978	.622
2	.473	.978	.622
3	.516	.948	.652
4	.559	.881	.719
5	.602	.799	.801
6	.645	.702	.898

C1: =1.6-B1 Menu

The formula `C1:=1.6-B1` is copied to the whole range `C2:C84` (see 3.4).

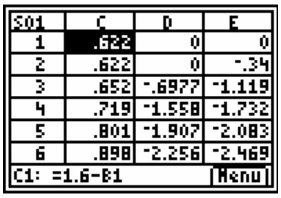
In column `D` we computed the velocity of the ball using the formula

`D2:=(B2-B1)/(A2-A1)`



S01	B	C	D
1	.978	.622	0
2	.978	.622	0
3	.948	.652	⁻.6977
4	.881	.719	⁻1.558
5	.799	.801	⁻1.907
6	.702	.898	⁻2.256

D3: =(B3-B2)/(A3-A2)



S01	C	D	E
1	.622	0	0
2	.622	0	⁻.34
3	.652	⁻.6977	⁻1.119
4	.719	⁻1.558	⁻1.732
5	.801	⁻1.907	⁻2.083
6	.898	⁻2.256	⁻2.469

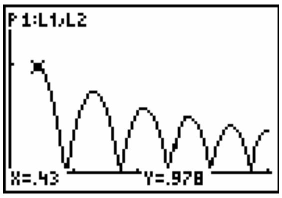
C1: =1.6-B1 Menu

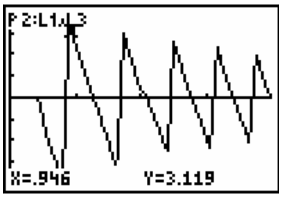
In column `E` you can see the velocity data computed by the CBR 2. Comparing columns `D` and `E` students realize that CBR 2 uses a different algorithm for computing velocity.

d. Finding the mathematical model

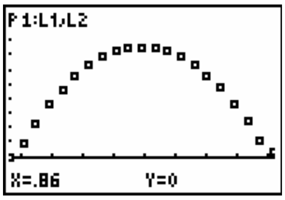
Looking at the time-distance diagram of the ball, you see that the ball first falls and then is reflected by the floor. Next it moves up, slowed down by gravity until it falls down again. This movement corresponds to repeated vertical throws. Therefore both phases of movement, i.e. up and down, can be described by quadratic functions. For this, the data for a complete bounce have to be selected from the total set of data. From this section of the graph (one bounce) the parameters for the ball's movement can be obtained.

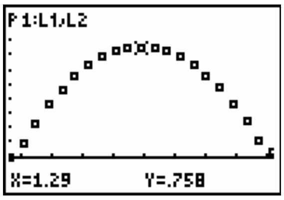
The values for height and velocity as a function of time are explored from several points of view.
The two figures beside show distance and velocity as functions of time.

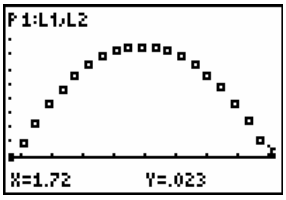




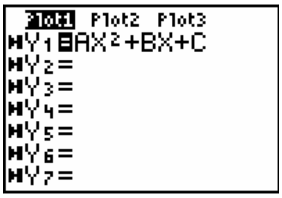
Optionally, a time-distance diagram or a time-velocity can be shown on the screen. When looking at the diagram – using trace – each point can be marked with the cursor and its coordinates can be seen on the bottom of the screen. This way the movement of the ball can be discussed with the class. The figures on the right show the time-distance diagram for the first complete bounce at different time values.



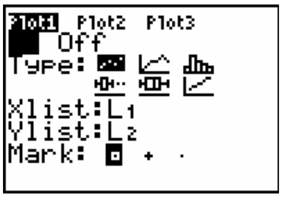




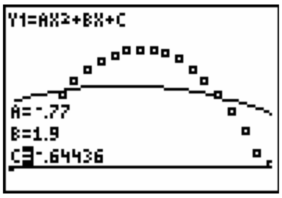
This looks like a parabola. With Transformation Graphing we can look for values of the parameters a, b and c in the formula $y = a(x-b)^2 + c$. In this formula you can easily see the maximum value and the corresponding time. Therefore we already know b and c (we can read them from the graph) and we store them to the calculator. For investigation the `Y=` screen is used. `L1` and `L2` are graphed as a scatter plot. With the cursor keys the value of a can be adjusted.



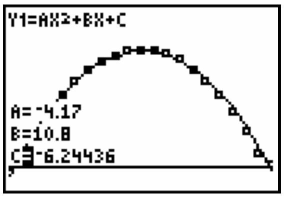
```
 Plot1 Plot2 Plot3
\Y1=AX²+BX+C
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



```
Plot1 Plot2 Plot3
On Off
Type:
Xlist:L1
Ylist:L2
Mark: ▫ + ·
```

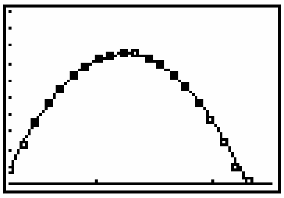


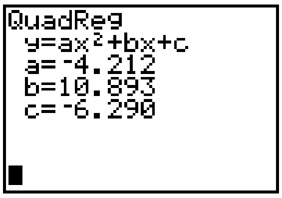
```
Y1=AX²+BX+C
A=⁻.77
B=1.9
C=⁻.64436
```



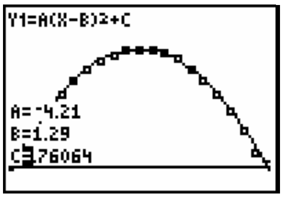
```
Y1=AX²+BX+C
A=⁻4.17
B=10.8
C=⁻6.24436
```

Another way to find the parameters is to do regression on the data in `L1` and `L2`.





```
QuadReg
 y=ax²+bx+c
 a=⁻4.212
 b=10.893
 c=⁻6.290
```



```
Y1=A(X-B)²+C
A=⁻4.21
B=1.29
C=.76064
```

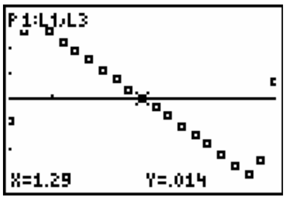
We can use Transformation Graphing again to find the values a, b and c of the parameters in $y = ax^2 + bx + c$ it is easier to $y = a(x-b)^2 + c$, because then we can read b and c from the graph.

e. Exploring velocity

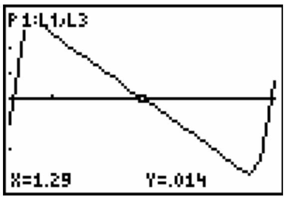
We stored the velocity (speed) in list `L3` as computed by the CBR 2. Graphing the velocity as a function of time (L1), gives a straight line. The diagrams below show the velocity-time graph in the interval of $x = 0.86$ to $x = 1.72$.

With Transformation Graphing we can look for values of the parameters a and b in the formula $y = ax + b$. For investigation the `Y=` screen is used.

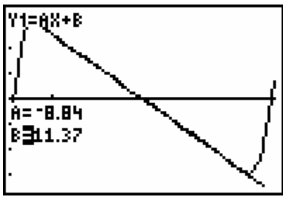
`L1` and `L3` are graphed as a scatter plot. With the cursor keys the values of a and b can be adjusted.



```
P1:L1,L3
X=1.29       Y=.014
```

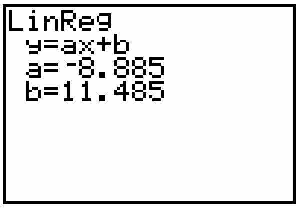


```
P1:L1,L3
X=1.29       Y=.014
```

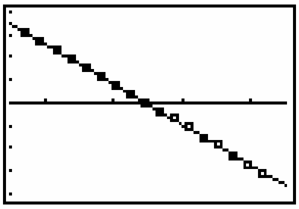


```
Y1=AX+B
A=⁻8.84
B=11.37
```

With regression for the time interval of $x = 0.86$ to $x = 1.72$ we get the linear function shown in the window below



```
LinReg
 y=ax+b
 a=⁻8.885
 b=11.485
```



.

In this way we get a constant acceleration of 8.89 m/s^2. This is significantly below the terrestrial gravity constant of 9.81 m/s^2.

It can be seen that the velocity, v, is 0 m/s at the start, decreasing to -3.7 m/s just before touching the floor. Then v rapidly increases to 0 and further up to 3.1 which is, comparing absolute values, lower than the v just before the rebound. With each next bounce energy is lost and finally the ball stops.

f. A few further questions

- Why is the experimentally obtained acceleration (calculated by the CBR 2) significantly smaller than g?
- Why can air friction not be the cause for a reduction of acceleration?
- Which additional force works against gravity when the ball is falling?
- What is the time course of potential and kinetic energy?
- What can be said about total energy?

When interpreting the time-distance graph, students will recognize the physical concept of movement caused by constant acceleration, the unavoidable transformation of kinetic energy in friction energy and also the mathematical representatives of individual graphs and their subsections. Each individual bounce is described by a convex parabola; its parameters are determined by the experiment and are interpreted physically. Students build mathematical models.

2. Boyle’s Law for Gas Pressure

a. Introduction

When a gas inside a closed container is compressed, its pressure and volume usually change. As the force exerted on the gas increases, the pressure increases while its volume decreases. Two quantities that change in this sort of way are said to vary inversely (inversely proportional). Even so both quantities may change, their product always stays the same.

If we suppose that x and y represent the quantities that are inversely related, then $xy = c$, where c represents a positive constant.

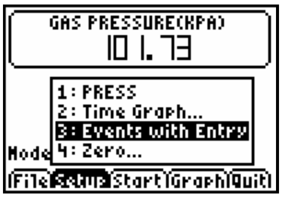
In this experiment students will explore the theory that pressure and volume vary inversely and will conclude with a formula that describes the special experiment and will investigate some questions like these:

- Could the volume ever be zero cc?
- Why or why not?
- What would be the corresponding pressure?

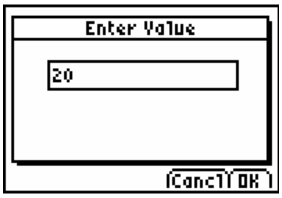
With the EasyLink adaptor it is very easy to connect sensors directly to the TI-84 Plus and perform real experiments. For example with the Gas Pressure Sensor you can investigate the relationship between volume and pressure of an amount of air in a syringe. The range for the Vernier pressure sensor is 0 to 210 kPa.

b. Performing and analyzing the experiments

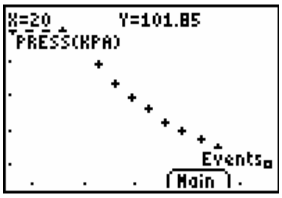
A volume of 20 ml is connected to the sensor. There is no extra pressure put on the air yet.

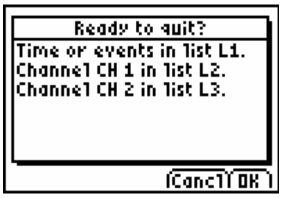

GAS PRESSURE(KPA)
101.73

```
1: PRESS
2: Time Graph...
3: Events with Entry
4: Zero...
```

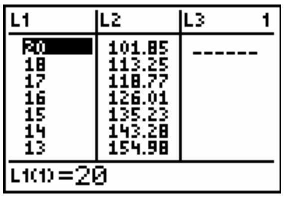

Enter Value

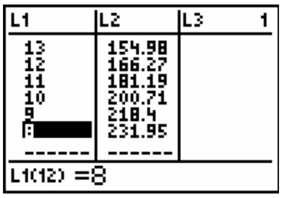
20

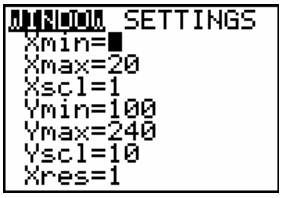


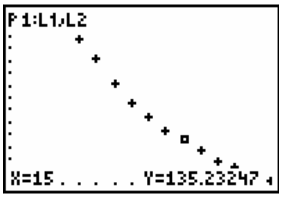

Ready to quit?

Time or events in list L1.
Channel CH 1 in list L2.
Channel CH 2 in list L3.


```
L1      L2      L3     1
20      101.85  ------
18      113.25
17      118.77
16      126.01
15      135.23
14      143.28
13      154.98
L1(1)=20
```

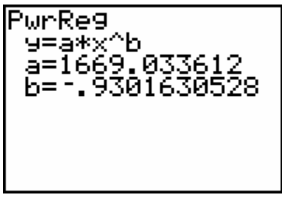

```
L1      L2      L3     1
13      154.98
12      166.27
11      181.19
10      200.71
9       218.4
8       231.95
------  ------
L1(12) =8
```

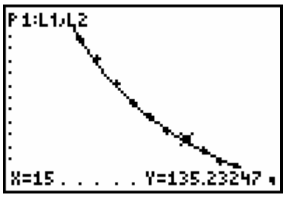

WINDOW SETTINGS
Xmin=
Xmax=20
Xscl=1
Ymin=100
Ymax=240
Yscl=10
Xres=1

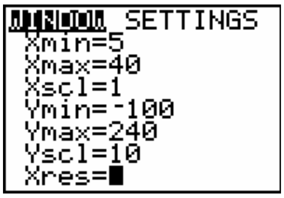


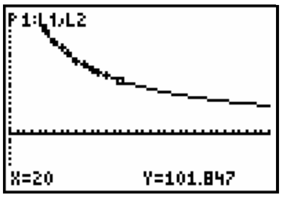
The starting values are volume 20 cc and pressure 101.73 kPa. Then the syringe is pressed, and for each of the following volumes, the pressure is measured and stored as events: 18, 17, 16, 15, 14, 13, 12, 11, 10, 9, 8.

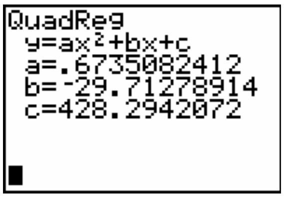
Now we are going to investigate what kind of relationship exists between volume and pressure. From the graph there are several possibilities: a quadratic function (a parabolic graph), an exponential function, a power function or a hyperbolic function (that is a special power function with exponent -1). With regression the best fit function can be found. Below are the results for power, quadratic and exponential regressions with their corresponding graphs.

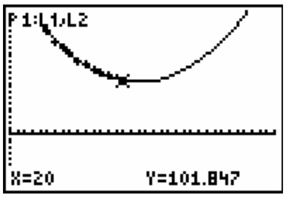

```
PwrReg
 y=a*x^b
 a=1669.033612
 b=-.9301630528
```



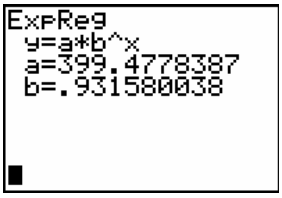

WINDOW SETTINGS
Xmin=5
Xmax=40
Xscl=1
Ymin=-100
Ymax=240
Yscl=10
Xres=

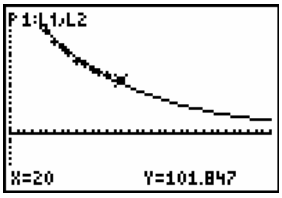



```
QuadReg
 y=ax²+bx+c
 a=.6735082412
 b=-29.71278914
 c=428.2942072
```



The quadratic equation fits the collected data very well, but what happens if the volume is increased again? Students have to investigate different “what-would-be-if” scenarios and have to decide what the right model is.

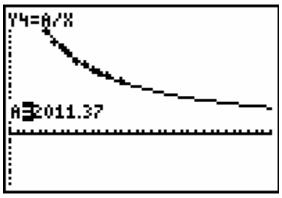

```
ExpReg
 y=a*b^x
 a=399.4778387
 b=.931580038
```

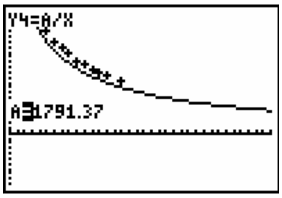


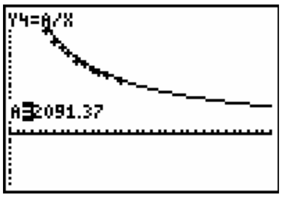
The exponential regression seems to fit the collected data very well too. Have students found the right model yet? What is the difference between the power function shown above and the exponential function shown on the left?

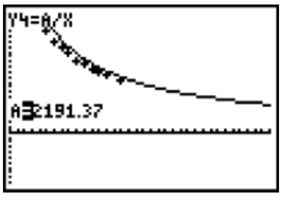
In the 17th century Boyle discovered that the relation between gas pressure, p, and volume, V, is $pV = \text{Constant}$. Based on our measurements we conclude that the constant in our example is about 2000 (for 20 ml the pressure was almost 102 kPa).

With Transformation Graphing we can easily explore the effects of different values of the constant on the graph and we conclude that 2011 fits best. See the screen shots below where the formula `y` = `A/X` is explored and the value for `A` is adjusted.









3. Newton’s law of cooling

a. Introduction

Students know from everyday life that hot tea cools to room temperature after some time. What determines the cooling process? Does the temperature decrease with a constant rate as shown in Figure 1 or is the decrease faster at the beginning and slower towards the end of cooling to room temperature as in Figure 2? Or is it slow in the beginning and fast towards the end as sketched in Figure 3?

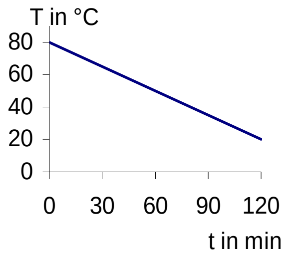


Figure 1

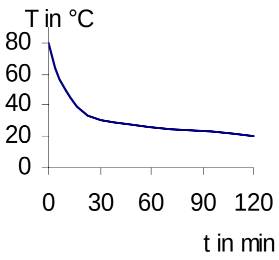


Figure 2

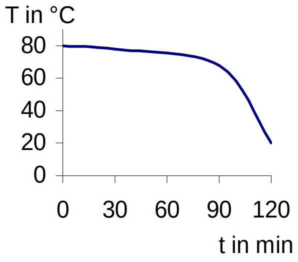
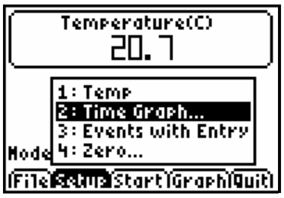


Figure 3

In this experiment students examine the cooling of hot water with the goal to create a model that describes the process.
They can also predict the time it takes for the hot water to cool to room temperature of 20.7°C (see the figure on the right).

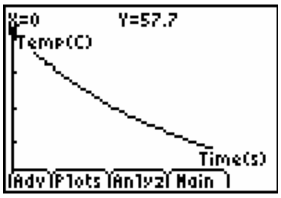


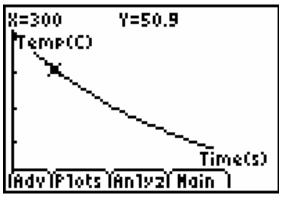
Isaac Newton modeled the cooling process assuming that the rate at which thermal energy moved from one body to another is proportional (by a constant k) to the difference in temperature between the two bodies, T_{diff}. From this simple assumption he showed that the temperature change is exponential in time and can be predicted by $T_{\text{diff}} = T_0 e^{-kt}$, where T_0 is the initial temperature difference.

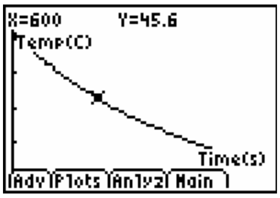
b. Performing the experiment

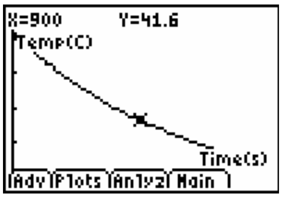
For this experiment we use a small quantity of hot water at a temperature of about 40°C above room temperature. EasyTemp can be connected directly to the USB port of the TI84 Plus. When EasyData is installed, it starts automatically and students can start collecting and analyzing data with their TI-84 Plus.

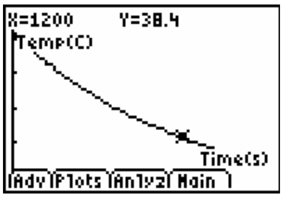
The graphical representation of the data collection in EasyData:

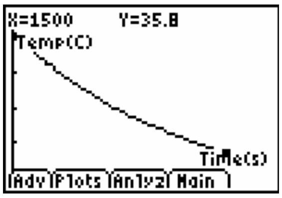

X=0 Y=57.7

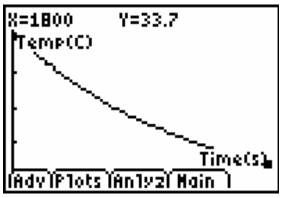

X=300 Y=50.9

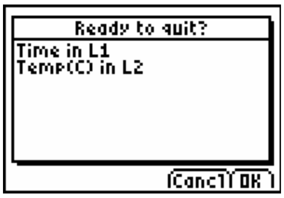

X=600 Y=45.6

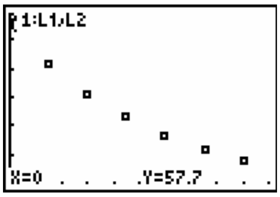

X=900 Y=41.6


X=1200 Y=38.4

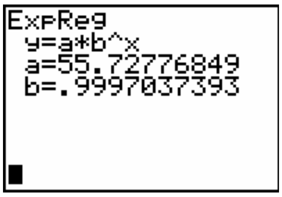

X=1500 Y=35.8

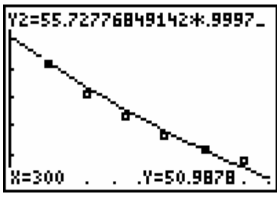

X=1800 Y=33.7

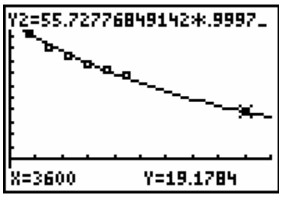

Ready to quit?
Time in L1
Temp(C) in L2

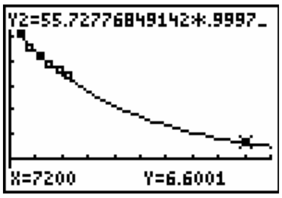

P1:L1,L2
X=0 Y=57.7

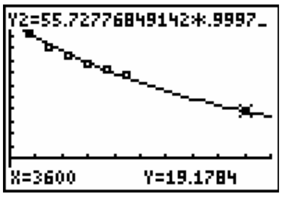
Now we are going to investigate what kind of relationship exists between time and temperature. The graph allows for various options: a quadratic function (a parabolic graph), an exponential function, a power function or a hyperbolic function (a power function with exponent -1). With regression the best fit function can be found. Below are the results for exponential regression with its corresponding graphs.

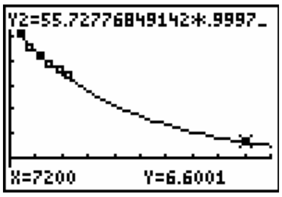

```
ExpReg
 y=a*b^x
 a=55.72776849
 b=.9997037393
```

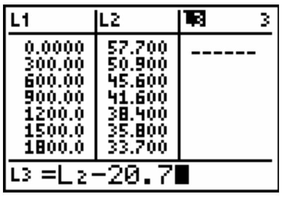

Y2=55.72776849142*.9997_
X=300 Y=50.9878

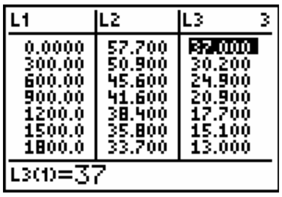

Y2=55.72776849142*.9997_
X=3600 Y=19.1784

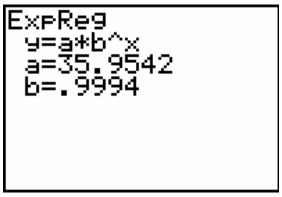

Y2=55.72776849142*.9997_
X=7200 Y=6.6001

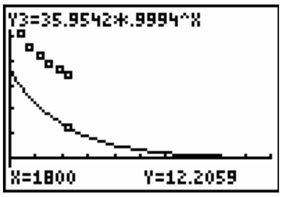

Y2=55.72776849142*.9997_
X=3600 Y=19.1784

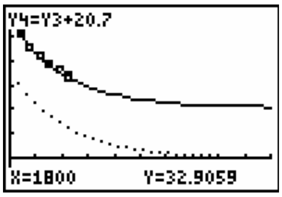

Y2=55.72776849142*.9997_
X=7200 Y=6.6001

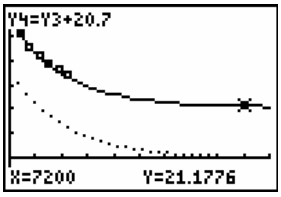

```
L1       L2       L3      3
 0.0000  57.700   ------
 300.00  50.900
 600.00  45.600
 900.00  41.600
 1200.0  38.400
 1500.0  35.800
 1800.0  33.700
L3 =L2-20.7
```


```
L1       L2       L3      3
 0.0000  57.700   37.000
 300.00  50.900   30.200
 600.00  45.600   24.900
 900.00  41.600   20.900
 1200.0  38.400   17.700
 1500.0  35.800   15.100
 1800.0  33.700   13.000
L3(1)=37
```

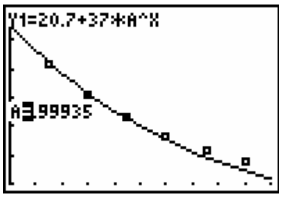

```
ExpReg
 y=a*b^x
 a=35.9542
 b=.9994
```

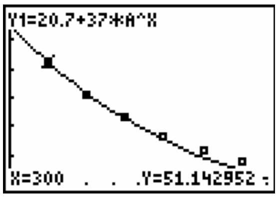

Y3=35.9542*.9994^X
X=1800 Y=12.2059

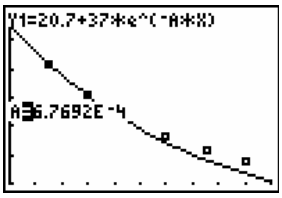

Y4=Y3+20.7
X=1800 Y=32.9059


Y4=Y3+20.7
X=7200 Y=21.1776

And with Transformation Graphing.


Y1=20.7+37*A^X
A=.99935


Y1=20.7+37*A^X
X=300 Y=51.142952


Y1=20.7+37*e^(-A*X)
A=6.7692E-4

c. Questions in the classroom

In the classroom the following questions can be asked.

- The curve describes the decrease of temperature over time. Analyze the curve and describe in your own words how the temperature decreases. Which of the curves best fits your results?
- The data collected during the experiment are stored in lists `L1` and `L2`. `L1` lists the time in seconds and `L2` the corresponding water temperatures. Plot the data points in a scatter diagram!
- Can you find a regression function that fits your temperature curve?
- Use the regression functions on the calculator to find the best fit, and plot the regression curves together with the recorded data.
- The room temperature T_R was 20.7°C. What could be wrong with the regression function you selected? What kind of "growth" do we have here? What are the properties the "right" regression function should have?
- The cooling down of a solution to the temperature T is described by Newton's law of cooling: $T = T_R + T_0 e^{-kt}$ or $T = T_R + T_0 \cdot a^t$ (time t in minutes).

T	temperature at time t
T_R	basic temperature (room temperature)
T_0	difference in temperature between the liquid and the room temperature at $t = 0$
a	constant, depending on the fluid's properties

- Try to fit the temperature function above to the actually measured curve by adjusting the parameters k and a. What is the relationship between a and k?
- Make a list `L3` with the temperature differences of measured temperature with room temperature. Now you can do exponential regression analysis. Use the function obtained and determine the values for T_0 and a! Then use the room temperature T_R to find an exponential function describing the decreasing temperature curve.
- Do the values for T_R, T_0 and a make sense to you? Why must a be smaller than 1? Explain in detail your explanation to these questions.
- Draw the graph of this function obtained together with the curve of the measured data (use coordinates). What do you think? Is this new exponential function a good approximation?
- Given the mathematical approximation for the measured temperature curve, how much time would it take for the water to cool down to room temperature?
- Has the starting temperature of the hot water any impact on the value obtained for a? Repeat the experiment with different starting temperatures to answer this question.
- What could you do to your experimental setting to decrease the value of k in another run? What quantity does k measure?
- If your starting temperature difference is cut in half, does it take half as long to get 1°C above room temperature? Why or why not does it take half as long?
- <u>A small research project</u>

  A coffee drinker is faced with the following dilemma. She is not going to drink her coffee with cream for ten minutes, but wants it still to be as hot as possible. Is it better to add immediately the room-temperature cream, stir the coffee, and let it sit for ten minutes, or is better to let the coffee sit for ten minutes and then add and stir in the cream? Use an EasyTemp and a calculator to examine this dilemma. Explain your results in terms of the assumptions Newton made about cooling. What changes, if the coffee drinker also wants to add sugar?

4. The pendulum

a. Introduction

If a pendulum (an object on a string) is pulled back and released, it will swing back and forth over time with a regular pattern occurring. It will eventually stop, but over a short time period the pendulum exhibits simple harmonic motion. This motion can be modeled with a periodic function.

In this activity students collect data from the motion of a pendulum. Then a periodic function is found that models the motion. Its parameters will be related to the time for one period, the distance it was pulled back and how far it is from the motion detector.

There are at least three things students could change in the pendulum that might effect the time for one complete cycle (called the period):

- the amplitude of the pendulum swing,
- the length of the pendulum, measured from the center of the pendulum to the point of support and
- the mass of the pendulum.

To investigate the pendulum students have to do a controlled experiment. They need to make measurements, changing only one variable at the time as a basic principle of scientific investigation. By conducting a series of controlled experiments with the pendulum, students can determine how each of these quantities effects the period, when they measure the period of a pendulum as a function of amplitude, as a function of length and as a function of mass.

The force driving the pendulum back to the equilibrium position is given by $F = mg\sin\varphi$.

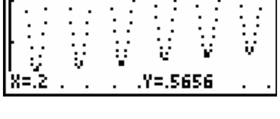
For small angles we can use $F = mg\frac{y}{l}$, where y is the distance from the starting point to equilibrium position and l is the length of the string.

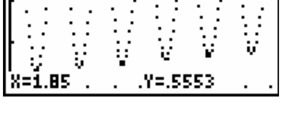
b. Performing the experiment and collecting the data

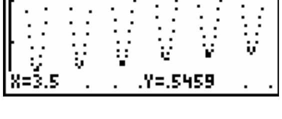
In this experiment we will use a motion detector (CBR 2) to plot the position vs. time graph for a simple pendulum. Students will use their data to find a formula that describes the position vs. time graph.

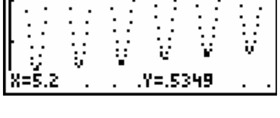
A string is tied to a mass (in the experiment we use a ball – see figure on the right). By trying different masses on the string, students will explore if the period of the pendulum depends on the mass and/or on the length of the string or on the amplitude, too. The mass is held from about 10° from vertical and then released.

The CBR 2 is aimed at the pendulum mass. Then the mass is pulled back about 10 centimeters and released, so that it swings toward and away from the sensor. The graph appears to be a sine or a cosine graph.





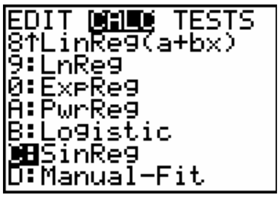


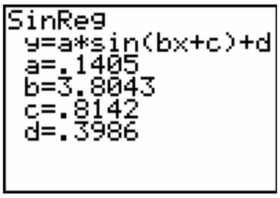


c. Looking at the results

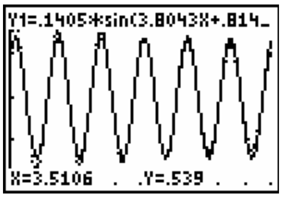
A pendulum completes a cycle as it moves from one extreme position to the other and back again. The time required for the pendulum to complete a cycle is called the period. To find the pendulum's period students have to trace to the first peak and record the time (x-value). Then they have to trace to the second peak and record the time value again. The period T is the difference between two time peaks (in the example T = 1.65 s).

With regression the best fit sine function can be found. The screens below show the result for sine regression with its corresponding graph.





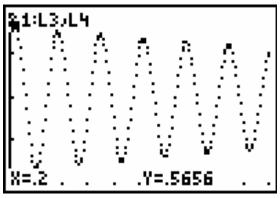
```
SinReg
 y=a*sin(bx+c)+d
 a=.1405
 b=3.8043
 c=.8142
 d=.3986
```

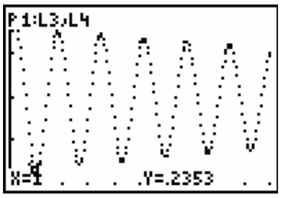


The position of the pendulum can be modeled with $y = a\sin(bx+c)+d$. In this formula y is the horizontal distance from the equilibrium position, a is the amplitude of the motion, b depends on the frequency of the oscillation ($b = \frac{2\pi}{T} \approx 3.81$), x is the time and c is a phase constant. The average between the maximum and the minimum values is the vertical shift d.

It can be found by adding the maximum and the minimum together and dividing by two. You get the distance from the CBR 2 to the pendulum's rest position.

The distance from the maximum to the minimum is twice the amplitude. To find the amplitude, we subtract the minimum from the maximum and then divide by two. You get the value to the distance from the pendulum's rest position to the point it was pulled back.

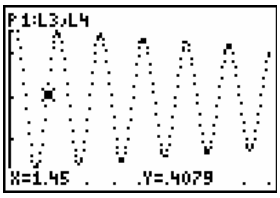




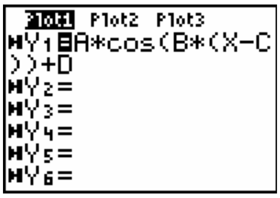
The formula $y = a\sin(bx+c)+d$ used by regression on the calculator can be more difficult to students than the form $y = a\sin(b(x-c))+d$ or $y = a\cos(b(x-c))+d$.
If the graph is modeled with the cosine function the horizontal shift is easier to identify than the horizontal shift of the sine curve. The phase shift for a cosine curve would be the time at which the first maximum occurs.

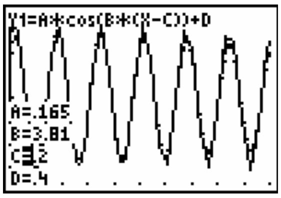
In this activity we use the sine curve $y = a\sin(bx+c)+d$ created by regression. Sine and cosine graphs only differ by a horizontal shift.

The phase shift will be the x-value of the point halfway between the minimum and the next maximum. The y-value will correspond to the vertical shift. Therefore students can trace to the point, where the y-value is most nearly the value of d and record the x-value as e. They get c by evaluating $c = -(be - 2\pi)$.



With Transformation Graphing we can look for values of the parameters `A`, `B`, `C` and `D` in the formula $y = A\cos(B(x-C))+D$. For investigation the `Y=` screen is used. With the cursor keys the values of `A`, `B`, `C` and `D` can be adjusted to find a graph that best fits the scatter plot.


Plot1 Plot2 Plot3
Y1=A*cos(B*(X-C
))+D
Y2=
Y3=
Y4=
Y5=
Y6=



`A` is the amplitude. That distance from the maximum to the minimum is twice the amplitude.

`A` = (0.5656 -0.2353)/2 ≈ 0.165

The vertical shift `D` is the average between the maximum and the minimum values.

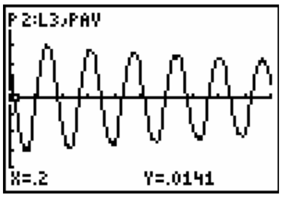
`D` = (0.5656 + 0.2353)/2 ≈ 0.400

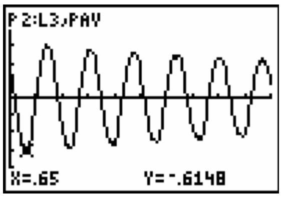
The difference in time between the first two maximum values is the period T.

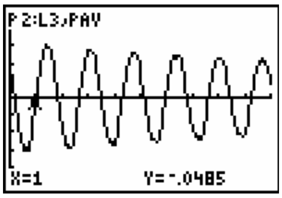
We get `B` by evaluating $\mathrm{B} = \frac{2\cdot\pi}{T} \approx 3.81$.

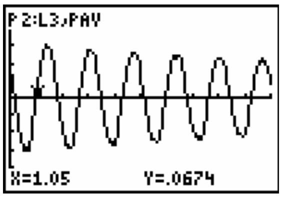
The phase shift `C` for a cosine curve is the time at which the first maximum occurs.

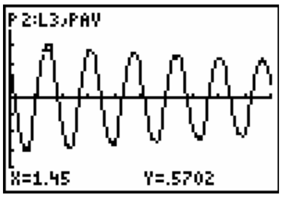
d. Modeling velocity of the pendulum

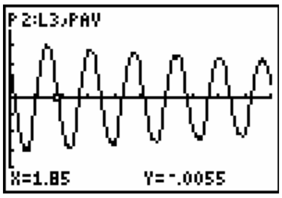


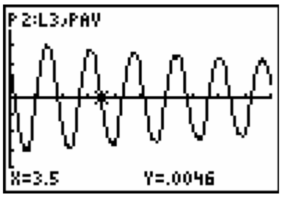


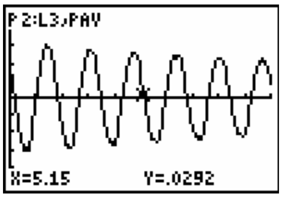




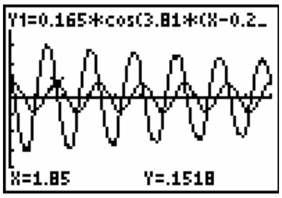


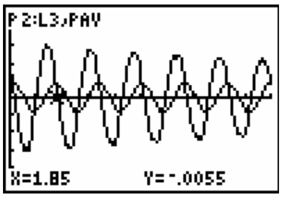




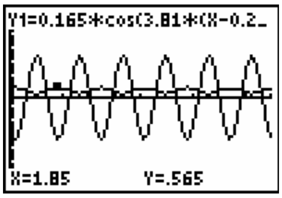


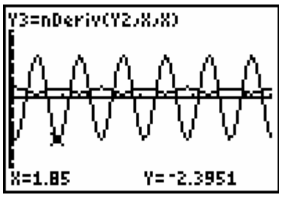
The period of the displacement and velocity are equal. When the distance is at a maximum, the velocity is zero. This makes sense because the mass stops as it turns around to change direction. The mass moves the fastest as it passes through the equilibrium position.





The period of the acceleration is the same as the period for the distance plot. When the distance is maximum, the acceleration is minimum. The acceleration is zero when the mass passes through the equilibrium position.





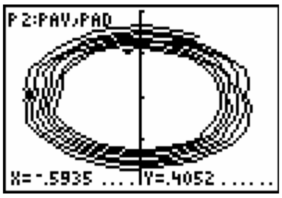
e. Extensions and Questions in the classroom

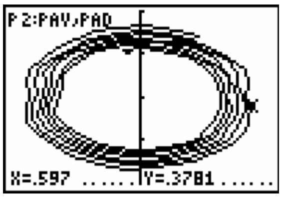
Once students have a formula for the position vs. time graph of the pendulum motion, they can take the derivative of the formula. This represents the velocity of the pendulum at any time t. The derivative of velocity is acceleration.

- How does the velocity vs. time graph compare with the position vs. time graph?
- When during the pendulum motion is the velocity zero?
- When is the velocity maximum?
- Describe the position and velocity when the acceleration is maximum. Do the same when the acceleration is zero.
- Give a general description of the pendulum’s position, velocity and acceleration when the pendulum mass is passing through the at-rest position and when it is farthest from the detector.
- Determine how the period depends on amplitude. Measure the period for five different amplitudes.
- Investigate the effect of changing pendulum length on the period.
- Determine if the period is effected by changing the mass. Does the period appear to depend on the length of the pendulum (the string attached to the ball)? Do you have enough data to answer this for sure?
- To examine more carefully how the period T depends on the pendulum length l, you can create two additional graphs of the same data: T^2 vs. length and T vs. length^2.

  Using Newton’s law, you could show that for a simple pendulum the period T is related to the length l and free fall acceleration g by $T = 2\pi\sqrt{\frac{l}{g}}$ or $T^2 = (\frac{4\pi^2}{g})l$.

  Does one of the graphs support this relationship?
- Determine a value for g from your graph T^2 vs. length.
- Try a larger range of amplitudes. What can you find for large amplitudes?
- The two figures below show distance vs. velocity graphs of the pendulum motion. Explain what you can see there and how the figures would look like if the motion wasn’t damped.





5. A mass on a spring – a further simple harmonic motion

a. Introduction

A mass hanging on a spring is a simple system that can be put to vibration. The force applied on an ideal spring is proportional to how much it is stretched or compressed. Given this proportional force, the up and down motion of the mass is called simple harmonic and the position can be modeled with $y = A\cos(2\pi f t + \varphi)$. In this formula, y is the vertical distance from the equilibrium position, A is the amplitude of the motion, f is the frequency of the oscillation, t is the time and φ is the phase constant.

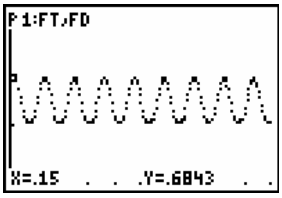
The following experiment will help to clarify each of these terms and describe harmonic motion with the help of mathematical functions. Students measure the position and velocity as a function of time for a vibrating mass and spring system. After determining the amplitude, period and phase constant of the observed motion of a mass and spring system they compare their data to a mathematical model of simple harmonic motion.

b. Performing the experiment and collecting the data

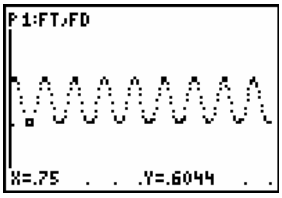
A spring is attached to a horizontal rod connected to a ring stand and a mass hangs on the spring. The motion detector is placed at least 75 cm below the mass.

Then the mass is lifted upward five to ten centimeters and then released. It should vibrate along a vertical line only and should never come closer than 40 cm to the motion detector (in this experiment CBR 2 is used, directly connected to the calculator).

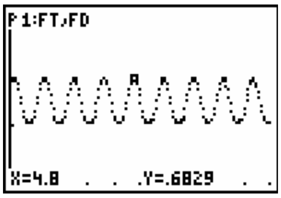
The distance graph should show a clean sine curve. For these data the period T of the motion can be calculated by (4.8 – 0.15)/4 = 1.2 s.



```
X=.15 . . .Y=.6843 . .
```



```
X=.75 . . .Y=.6044 . .
```



```
X=4.8 . . .Y=.6829 . .
```

The frequency f is the reciprocal of the period, $f = \frac{1}{T}$.

Based on the experiment the frequency is calculated as 0.83 Hz.

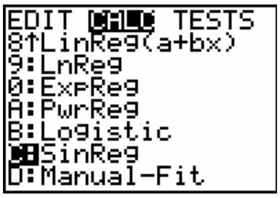
The amplitude, A, of simple harmonic motion is the maximum distance from the equilibrium position. A can be calculated as (0.6843-0.6044)/2 = 0.04 m.

c. Looking at the results

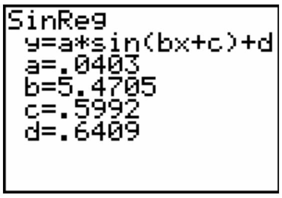
With regression the best fit sine function can be found. The position of the mass can be modeled with $y = a\sin(bx + c) + d$.

In this formula y is the vertical distance from the equilibrium position, a is the amplitude of the motion, b depends on the frequency of the oscillation ($b = \frac{2\pi}{T} \approx 3.81$), x is the time and c is a phase constant.

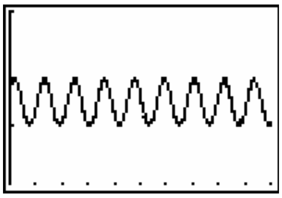
The average between the maximum and the minimum values is the vertical shift d. It can be found by adding the maximum and the minimum together and dividing by two. You get the distance from the CBR 2 to the rest position of the mass on the spring.



```
EDIT CALC TESTS
8↑LinReg(a+bx)
9:LnReg
0:ExpReg
A:PwrReg
B:Logistic
C:SinReg
D:Manual-Fit
```

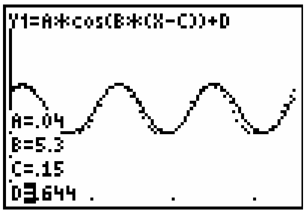


```
SinReg
 y=a*sin(bx+c)+d
 a=.0403
 b=5.4705
 c=.5992
 d=.6409
```



Experimental data can be compared to the sine function model using the formula entered in your calculator. The model formula in the introduction, which is similar to the one in many textbooks, gives the distance from the equilibrium position of the mass. The CBR 2 reports the distance from the detector. To compare the model to the experimental data the equilibrium distance to the model has to be added.

With Transformation Graphing we can look for values of the parameters `A`, `B`, `C` and `D` in the formula $y = A\cos(B(x - C)) + D$, which is much easier. The `Y=` screen is used for investigation and with the cursor keys the values of `A`, `B`, `C` and `D` can be adjusted to get the graph that best fits the scatter plot.



```
Y1=A*cos(B*(X-C))+D
A=.04
B=5.3
C=.15
D=.644
```

`A` is the amplitude – it is the maximum distance from the equilibrium position. That distance from the maximum to the minimum is twice the amplitude: `A` = (0.6843 – 0.6044) / 2 $\approx$ 0.04 m.

The vertical shift `D` is the average between the maximum and the minimum values:
`D` = (0.6843 + 0.6044) / 2 $\approx$ 0.644.

The difference in time between the first two maximum values is the period T. We get `B` by this formula: $\mathtt{B} = \frac{2\pi}{T} \approx 5.236$.

The phase shift `C` for a cosine curve is the time at which the first maximum occurs.

d. Further Questions and extensions

- Compare the position-time graph and the velocity-time graph. How are they the same? How are they different?
- Trace the velocity graph to view the data values. Record a time when the velocity is maximized and another time when the velocity is zero. Then record the position of the mass at these times. Where is the mass, when the velocity is zero, relative to the equilibrium position? Where is the mass, when the velocity is maximized?
- Predict what would happen to the plot of the model if you doubled the parameter `A` (the amplitude).
- Similarly, predict how the model plot would change if you doubled f and then check by modifying the model definition.
- Does the frequency, f, depend on the amplitude of the motion? Try to get enough data to draw a firm conclusion.
- Does the frequency, f, depend on the mass used? Try to get enough data to draw a firm conclusion.
- Investigate how changing the spring amplitude changes the period of motion.
- How will damping change the data? Tape a card to the bottom of the mass and collect additional data for more than 10 seconds. Does the model still fit well in this new situation?
- Do additional experiments to discover the relationship between the mass and the period of motion.