4. Parameter

In this unit you will discover how an 'extra letter' or parameter in a formula may lead to studying families of curves or dynamic curves instead of one static graph. The unit consists of three sub-units, entitled The shifting parabola, Rotating lines, and Lines touching a parabola.

4.1 Shifting parabola

Goal of the task

This task is part of the Parameter Unit, which includes the tasks ‘The shifting parabola’, ‘Rotating lines’, and ‘Lines touching a parabola’.

The goal of this task is that students realise that the value of the parameter, in this case c, determines the position and/or the form of the graph. Furthermore, the task aims to help students understand that an additional criterion concerning the distance of the zeros ‘filters out’ one or more parameter values. The first part of the task concerns the effect that parameter changes have on the parabola which contains this parameter in its equation. The second question requires students to identify the parameter value that fulfils a condition concerning the zeros of the parabola. The final question requires a mental shift from students as the parameter takes the character of an unknown.

Students can approach these tasks graphically and the TII slider bar allows for a dynamic view of the situation. However, the aim is that students support their findings using algebra and the algebraic features of TII can be useful.

Target group and required time

The target group for this task are tenth-grade students of average to high ability in mathematics. Eventually, ninth-grade might be considered. The task requires approximately two lessons in the computer lab.

Preliminary TI InterActive! skills

The students need basic skills in using TII, in particular those related to working with graphs (trace) and for applying commands from the algebra menu such as ‘solve’ and ‘approximate’.

Preliminary mathematical skills

The required preliminary mathematical knowledge is limited to a basic knowledge of parabolas, in particular the role of the parameter *a* in $y = a \cdot x^2 + b \cdot x + c$. The task might fit well after a chapter in which parabolas have been taught.

File organization

The task consists of the following linked TII files:

- ParabNotebook.tii: The file in which the student writes the solutions and also the starting point which contains hyperlinks to the other files.
- ParabMain_Task.tii: The file that describes the main task.
- ParabHintA.tii: A file that suggests a question to start with if the student doesn’t know what to do with the main task A.

- ParabHintB.tii: A file that suggests a question to start with if the student doesn't know what to do with the main task B.
- ParabControlA.tii: Contains control questions for the main tasks, in order to provoke some reflection on the results.
- ParabAdditional.tii: Contains an additional task for the brighter / faster students.
- ParabSolution.tii: Contains the solutions.

The hyperlinks only work if the files are installed in the map c:\TII\Parameter\ShiftPar.
The file StartParameter in c:\TII\Parameter provides access to the three notebook files of the three units that together form the Parameter Unit.

Classroom organisation

We suggest a short plenary at the start of about 10 minutes to introduce the problem and demonstrate some important TII skills. The slider bar can be demonstrated, as well as some other TII skills such as the solve command. The equation of a parabola might need some practice.

Then students can work in pairs and explore the problem by substituting different values for the parameter c. By means of the slider bar in HintA students can get a more dynamic view of the situation and this enables them to develop an understanding of the relationship between the parameter value and the form of the parabola, at least at a phenomenological level.

The second lesson can also start with a short plenary demonstration or classroom discussion so that the findings of the first lesson are summarised. Eventually, a pair of students may be requested to demonstrate their findings to date.

Technical hints

Students start with opening the notebook file in TII. They then open the main task in a separate window using the hyperlink. Two versions of TII are now activated. By means of a right mouse click in the Windows menu bar the option 'Windows Cascade' offers opportunities for switching easily between the files. However, if the same hyperlink is used twice two versions of the same file will be opened! This may be a source of confusion and therefore requires some attention by the teacher.
As TII only supports absolute hyperlinks and no relative links, the links need to be adjusted after installation of the files, for example on the school network.

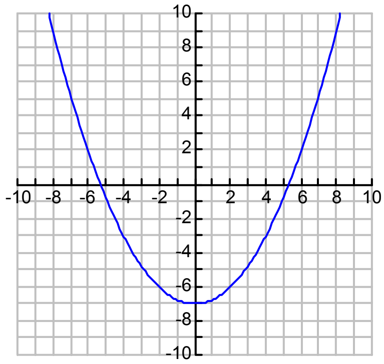
Didactical suggestions

Many students will see that the parabola narrows, but may not be able to explain this from the formula. To focus students' view on the vertical movement the hintA graph contains a point with a fixed x-coordinate.

Many students will describe the dynamics as 'the smaller the value of c, the bigger the distance between the zeros'. They probably will not search for algebraic expressions for the zeros. A suggestion by the teacher may be needed. There is also a difficulty in what students perceive as 'an exact answer'. It may be that students will say that 1.41 is more exact than $\sqrt{2}$, because it has more decimals. This is a useful issue for classroom discussion.

We consider graphs of the function *f* that is defined by: define $f(x) = \frac{x^2}{c} - 7$.

The graph of *f* for the case where the value of *c* equals 4 is shown below:



Main task

a. How does the value of *c* influence the form of the parabola? **Hint 1**

Explain this using the formula.

b. The value of *c* also affects the position of the zeros. **Hint 2**
For what value of *c* is the distance between the zeros exactly 6 units?

Control task

Please read the three descriptions below.

(i) If the value of *c* gets bigger, the parabola is pushed outwards.

(ii) If the value of *c* gets bigger, the parabola is pulled towards the *x*-axis.

(iii) If the value of *c* gets bigger, the parabola is pulled towards the line $y = -7$.

Use the fomula to explain which of the above descriptions you think most properly describes the situation.

Did you find an exact answer (a fraction)? If not, find out where you approximated and find a means to calculate the exact answer too.

Additional task

a. For which value of *c* is the distance between the two zeros equal to 50000 units?
Draw the graph of *f* for this particular case.

b. We now consider the graphs of the function: Define $g(x) = \frac{x^2}{4} - c$.

Investigate the effect of the value of *c* on the parabola.
Calculate the exact value of *c* so that the distance between the zeros is exactly 10 units.

4.2 Rotating line

Goal of the task

This task is part of the Parameter Unit, which includes the tasks 'The shifting parabola', 'Rotating lines', and 'Lines touching a parabola'. In this task, the Rotating Lines, which follows the shifting parabola task , students gain an understanding of a 'sliding parameter' and its effect on lines and their intersection points.

The initial focus for this task is that students study the effect of changing a parameter on lines that contain this parameter in their equation. Students are then required to identify parameter values that fulfil specific conditions. Finally students need to make a mental shift as the parameter takes on the characteristics of an unknown.

Students can approach these tasks graphically with the TII slider bar allowing for a dynamic view of the situation. However, the aim is that students also support their findings using algebra and therefore the algebraic features of TII can be useful.

The main task consists of two questions, one which considers the influence of parameters on the intersection point of two lines and the other which requires students to find values of the parameter which result in an intersection point to the right of the vertical line with equation $x = 7$.

Target group and required time

The target group for this task is tenth-grade students who are of average to high ability in mathematics. Eventually, ninth-grade might be considered.
The task requires approximately two lessons in the computer lab.

Preliminary TI InterActive! skills

Students need some basic skills in using TII, in particular for using graphs and for applying algebraic commands such as the solve command.

Preliminary mathematical skills

The required preliminary mathematical knowledge is limited to a basic knowledge of linear functions.

File organization

The task consists of the following linked TII files:

- RotLineNotebook.tii: The file where the student writes the solutions and this is also the starting point as it contains links to the other files.
- RotLineMain_Task.tii: The file that briefly describes the task.
- RotlineHintA.tii: A file that suggests a question to start with if the student doesn't know what to do with the main task.
- RotlineHintB.tii: A file that suggests a question to start with if the student doesn't know how to proceed further after hintA.
- RotlineControlA.tii: Contains a control question on the main tasks, in order to provoke some reflection on the results.
- RotLineAdditional.tii: Contains an additional task for the brighter / faster students.
- RotLineSolution.tii: Contains the solutions.

The hyperlinks only work if the files are installed in the map c:\TII\Parameter\RotLine. The file StartParameter in c:\TII\Parameter provides access to the three notebook files of the three units that together form the Parameter Unit.

Classroom organisation

This task can be done after 'the sliding parabola'. We suggest that teachers start the session with a short plenary of about 10 minutes to introduces the problem and to demonstrate some important TII skills.

Students can work in pairs to explore the problem by substituting different values for the parameter *c*. By means of the slider bar in HintA students will have a dynamic view of the situation. This will allow students to develop an understanding of the relationship between the parameter value and the lines, at least at a phenomenological level.

The second lesson can also start with a short plenary demonstration or classroom discussion so that the findings of the first lesson are summarised. The teacher may have a pair of students demonstrate their findings.

Technical hints

Students start with opening the notebook file in TII. They then open the main task in a separate window using the hyperlink. Two versions of TII are now activated. By means of a right mouse click in the Windows menu bar the option 'Windows Cascade' offers opportunities for switching easily between the files. However, if the same hyperlink is used twice two versions of the same file will be opened! This may be a source of confusion and therefore requires some attention by the teacher.
As TII only supports absolute hyperlinks and no relative links, the links need to be adjusted after installation of the files, for example on the school network.

Didactical suggestions

Most students are happy with the numerical answers to the task which can be found graphically. The task of the teacher is to suggest an algebraic approach, by means of questioning the certainty and exactness of the approximated results.

In our experience the control question in RotLineControl part B was not often answered by the students. This is something to keep track of as this question is intended to guide students towards an algebraic approach.

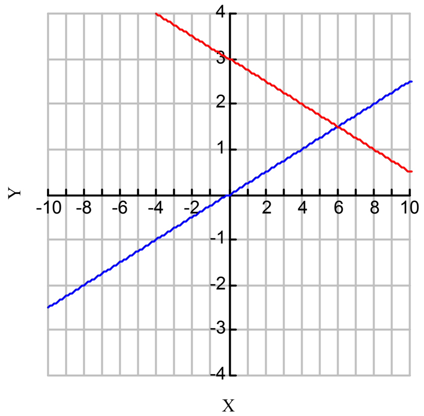
intro

We consider graphs of functions *f* and *g* that are defined by:

define $f(x) = c \cdot x$

define $g(x) = 3 - c \cdot x$

Below you see the graphs of *f* and *g* in case the value of *c* equals 1/4.



task

Main task

a. What is the effect of the value of *c* on the positions of the intersection of the two lines? Explain by means of the formulas. **Hint 3**

b. What values of *c* result in the intersection point being to the right of the line $x = 7$? Give an exact answer. **Hint 4**

task

Control task

a. Explain what happens to the intersection point when the value of *c* is negative.

b. You will probably have noticed that the value of *c* does not affect the height of the intersection point. Explain why this is the case using the formulas.

c. Did you find an exact answer (a fraction)? If not, then please find one.

d. Did you consider in your answer the possibility that the value of *c* is smaller than zero?

task

Additional task

a. What values of *c* lead to an intersection point that is on the left of the line $x = 1/100$? Provide an exact answer.

b. Find two linear functions which both contain a parameter *c* in their formulas and have the following property: If the value of *c* gets smaller, the intersection point of the two graphs goes straight upwards.

4.3 Lines touching a parabola

Goal of the task

This task is part of the Parameter Unit which includes the tasks ‘The shifting parabola’, ‘Rotating lines’ and ‘Lines touching a parabola’.

Following from tasks on the shifting parabola and the rotating lines this unit integrates the skills, experiences and conceptual understanding that were developed. This task aims to deal with parameters in a flexible and versatile way in order to solve a ‘difficult’ problem. For this task not many hints are provided to guide the student, but instead the aim is for students to use their experience to find their own way through the problem. Students are supposed to develop a problem solving strategy and carry out their plan to solve the problem.

The central focus of the task is to find lines through the point (0,-5) that touch the parabola with equations $y = x^2 - 3x - 2$. Please note that the students of the target group are not familiar with derivatives so they cannot use this in their strategy.

The TII slider bar allows for a dynamic view of the situation and an estimation of the right slope. It does accurate and quick graphing for the students allowing them to visualise the problem and convincing students that there is a solution. However, the exact answer will not be found using this approach and the task aims to have students find the exact answer using algebra. Therefore the algebraic power of TII which guarantees correct performance of hard calculations is useful.
After this main task the additional task presents an even harder but more fascinating problem. The additional task, ‘how to find the equation of a line that touches two given parabolas’ is borrowed from Luc Trouche (see reference below).

Target group and required time

The target group for this task are tenth-grade students who are of average to high ability for mathematics. Eventually, eleventh-grade might be considered.

The task requires approximately 4 to 7 lessons in the computerlab, depending on whether the additional task is attempted.

Preliminary TI InterActive! skills

The students need some basic skills in using TII for using graphs and for applying algebraic commands such as the solve command.

Preliminary mathematical skills

The required preliminary mathematical knowledge concerns some experience with using a parameter to represent a sheaf of lines, and with solving quadratic equations. Knowledge of differential calculus is not expected.

File organization

The task consists of the following linked TII files:

- LineParNotebook.tii: The file in which the student writes solutions and the starting point containing hyperlinks to the other files.
- LineParMainTask.tii: The file that briefly describes the task.
- LineParHint.tii: A file that suggests a question to start with if the student doesn’t know what to do with the main task.
- LineParControl.tii: Contains a control question on the task in hintA, in order to provoke some reflection on the results.

- LineParAdditional.tii: Contains an additional task for the brighter / faster students.
- LineParSolution.tii: Contains the solutions.

The hyperlinks only work if the files are installed in the map c:\TII\Parameter\LinePar.
The file StartParameter in c:\TII\Parameter provides access to the three notebook files of the three units that together form the Parameter Unit.

Classroom organisation

This task can be done after 'the shifting parabola' and the 'rotating lines'. We suggest that teachers start the session with a short plenary of about 10 minutes, to introduce the problem and reinforce some important TII skills. In particular, the concept of 'touching' lines or of tangent lines may need some illustration as students are not yet familiar with differential calculus. Also, it may be worthwhile to remind students that intersection points can be found with the solve command.

Students can then work in pairs to explore the problem graphically or by substituting different values for the parameter c. By means of the slider bar in the Hint-file students have a dynamic view of the situation. This will allow them to develop an understanding of the relationship between the parameter value and the lines, at least at a phenomenological level. The teacher, while discussing with pairs of students, can play 'the devil's advocate' to make sure that the students understand what they have to do and why.

The second lesson can also start with a short plenary demonstration or classroom discussion so to summarise the findings of the first lesson. The teacher may also have a pair of students demonstrate their findings.

Technical hints

Students start by opening the notebook file in TII. It is recommended that the main task is opened in a separate window so that two versions of TII are activated. By means of a right mouse click in the Windows menu bar the option 'Windows Cascade' offers opportunities for switching easily between the files. However, if the same hyperlink is used twice two versions of the same file will be opened! This may be a source of confusion and therefore requires some attention by the teacher.
As TII only supports absolute hyperlinks and no relative links, the links need to be adjusted after installation of the files, for example on the school network.

Didactical suggestions

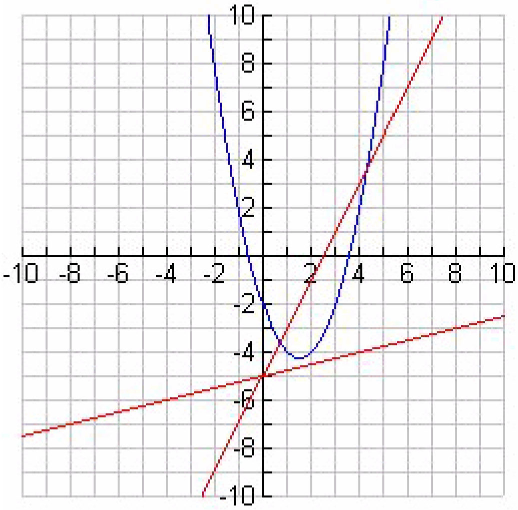
Most students may be happy with the numerical answers for the task which were found graphically. The task of the teacher is to suggest an algebraic approach through questioning the certainty and exactness of the approximated results.
The additional task is 'the ultimate challenge' that would be a shame to miss.

Reference

Trouche L. (1998). *Expérimenter et prouver, 38 variations sur un thème imposé.* [Experiment and prove, 38 variations to an imposed theme.] Montpellier, France: IREM, Université Montpellier II.

intro

Below is a coordinate plane with a parabola and two lines.



The parabola is the graph of the function *f*: define $f(x) = x^2 - 3 \cdot x - 2$.

Both lines go through the point (0 , -5). One line intersects the parabola at two points while the other line does not intersect the parabola. This suggests that there is a line through (0,-5) that intersects the parabola at exactly one point. In this case we say that the line *touches* the parabola.

task

Main task

Find the exact equation of the line through (0,-5) that touches the parabola. **Hint 5**

task

Control task

a. Did you find two solutions for *a*? If not, find the second one. Draw the graph with the two 'touching' lines.

b. Did you find approximate solutions? If you did then find the exact solutions.

c. So far the lines we wanted to touch the parabola went through the point (0, -5). As a generalisation we now consider lines through the point (0, *b*).

Find the values of *b* for which there exists lines that touch the parabola with equation $y = x^2 - 3 \cdot x - 2$.

task

Additional task

The Ultimate Challenge

Below you see a coordinate plane with two parabolas and one line.

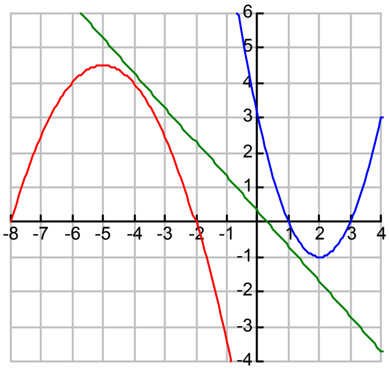
The parabolas are graphs of the functions *f* and *g*:

define $f(x) = x^2 - 4x + 3$

define $f(x) = \frac{-1}{2}x^2 - 5x - 8$

Find the exact equation of the line that touches both parabolas.

$$\text{define } l(x) = a \cdot x + b$$



$\text{solve}(f(x) = l(x), x)$ $\text{solve}(g(x)=l(x),x)$

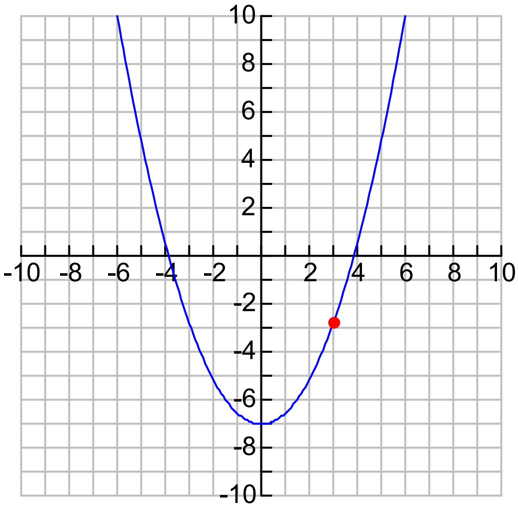
4.4 Hints

hint 1

We considered the function: define $f(x)=\frac{x^2}{c}-7$.

Below is the graph of *f*. Using the slider bar you can change the value of *c*.

$c = 2.1$



The folowing question may guide you towards the solution.

(i) Investigate what happens to the parabola when you change the value of *c*.

(ii) In the graph the point *A* with *x*-coordinate 3 is shown.

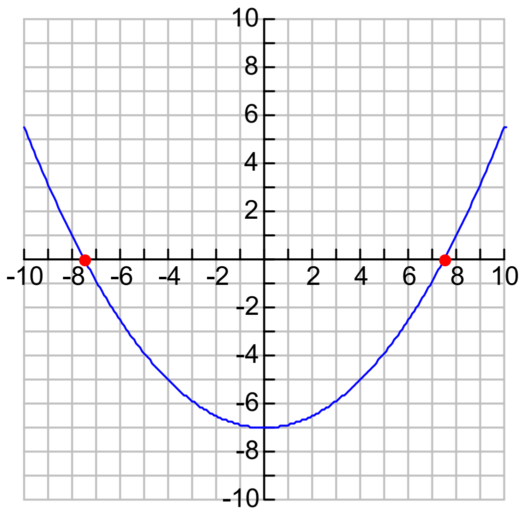
$A := [3. \quad f(3.)] = [3. \quad -2.71429]$

What happens to point *A* if the value of *c* changes?
Explain this using the formula for *f*.

(iii) There is one point that does not move. Which one? Explain why, using the formula for *f*.

hint 2

We considered the function: $f(x)=\frac{x^2}{c}-7$.



Alongside is the graph of *f*.

Using the slider bar you can change the value of *c*.

$c = 8$

The folowing questions may guide you towards the solution.

(i) Using the command **solve(** you can calculate the coordinates of the intersection point of the parabola with the *x*-axis. Consult the TII help if you don't remember the syntax for **solve(**.

(ii) Calculate the coordinates of the intersection point(s) of the parabola and the *x*-axis in general. The answer contains the letter *c*. Explain why.

(iii) Two solutions are found. State which solution corresponds to each intersection point.

(iv) The condition that the distance between the intersection points equals 6 units, implies that the zeros are $x = 3$ and $x = -3$. Find out how you can use the result of 2. to calculate the value of *c*.

hint 3

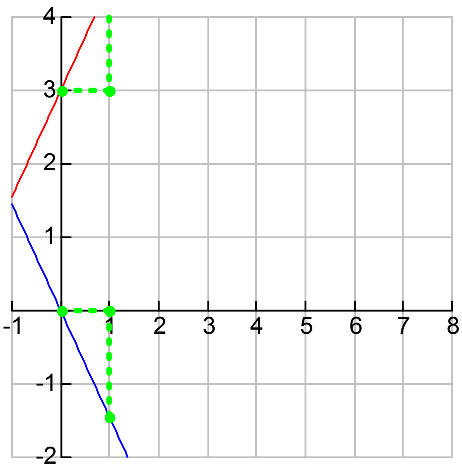
We have considered the following functions:

define $f(x) = c \cdot x$

define $g(x) = 3 - c \cdot x$

Below you see the graphs of *f* and *g*. Using the slider bar you can change the value of *c*.

$c = -1.45$



The following questions may guide you towards the solution.

(i) Investigate what happens with the lines and the intersection points when you change the value of *c*.

(ii) Complete: If the value of *c* gets smaller, the intersection point ...

(iii) Try to explain your findings using the formulas.
Think about what happens with the y-intercept of the lines and with the slopes.

hint 4

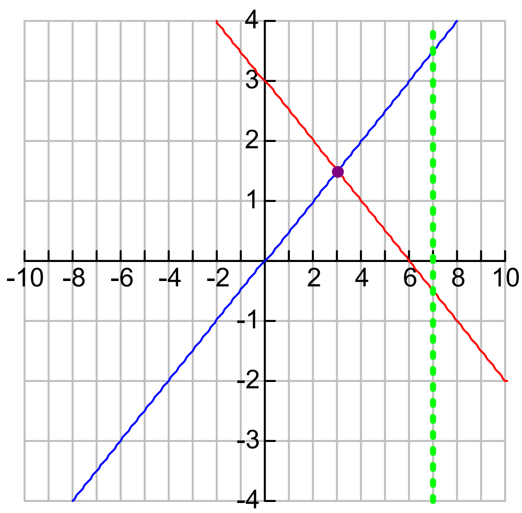
We have considered the following functions:

define $f(x) = c \cdot x$

define $g(x) = 3 - c \cdot x$

Alongside you see the graphs of *f* and *g*.
Using the slider bar you can change the value of *c*.

$c = .5$



The following questions may guide you towards the solution.

(i) Using the command **solve(** you can calculate the coordinates of the intersection point of the two lines. Consult the manual if you don't remember the syntax for **solve(** .

(ii) Calculate the coordinates of the intersection point in general. The answer contains the letter *c*. Explain why.

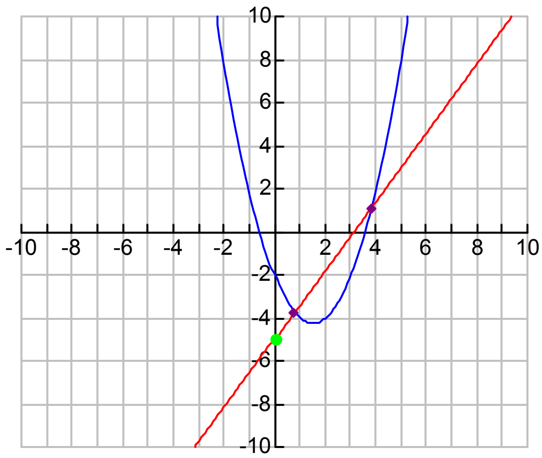
(iii) Find out how you could use the results from (ii) to answer the question when the intersection point is to the right of the line $x = 7$.

hint 5

The coordinate plane below shows the graphs of the functions *f* and *g*:

define $f(x) = x^2 - 3 \cdot x - 2$
define $g(x) = a \cdot x - 5$

The lines all go through the point (0,-5). (Think about how you can see that in the formula for *g*).
With the slider bar below you can change the value of *a* and thereby the slope of the line.



In the working below the *x*-coordinates of the intersection points of the graphs of *f* and *g* are calculated twice using the **solve** command, with the first calculation being for the value of *a* that is chosen in the slider bar. In the second calculation no value has been chosen for *a* and the solutions that are provided depend on *a*.

$\text{solve}(f(x) = g(x), x)$ $x = 3.81327 \text{ or } x = .786725$

$\text{delvar}(a)$

$\text{solve}(f(x) = g(x), x)$

$$x = \frac{-\left(\sqrt{a^2 + 6 \cdot a - 3} - a - 3\right)}{2} \text{ or } x = \frac{\sqrt{a^2 + 6 \cdot a - 3} + a + 3}{2}$$

The following questions may guide you towards the solution.

(i) Substitute $a = 1$ in the second set of solutions. Verify that this yields the same solutions as those obtained from the first method of solving.

(ii) Find a value of *a* for which the line and the parabola do not intersect. Substitute this value into the solutions obtained using the second method for solving. Explain why this does not yield a solution.

(iii) Use the general solution of the equation *f(x)* = *g(x)* to find out for which value of *a* there is exactly one intersection point.