3. Growth

Goal of the task

The Growth unit is divided into four sub-units: Linear Growth, Exponential Growth, Logistic Growth and Comparison.

Before starting with this unit on Growth, the students have been made familiar with exponential functions and logarithmic functions for about 8 -10 lessons. Mathematics in this unit, developed for a business college, is closely related to business and our aim is to teach our pupils applications of the mathematics, rather than the theory. One of the main topics in the students' second year of Mathematics is growth.

- The students should become familiar with the basic types of growth (linear, exponential, logistic).
- TII is used in notebook-classes only and has to be used in each lesson for calculating and exploring "the jungle".
- The worksheets are available on the internet for users of TII only and are intended for classroom usage. These worksheets consist of two kinds; one which features various situations and incorporates sliders and one which contains concrete tasks. The tasks are aimed at students of different levels.

The worksheets presented here do not replace the traditional course on growth, but must be seen as an extension or an addition. They do not represent a separate course on that topic.

Target group and required time

The target group of this unit is 10^{th} grade students with an interest in business applications of mathematical models. The Growth unit requires approximately 20 to 25 lessons, including practice time.

Preliminary TI InterActive! skills

The students need the basic skills in using TII which are addressed in the introductory unit of this booklet. Some basic skills in using Windows applications are also useful.

Preliminary mathematical skills

The required preliminary mathematical knowledge consists of a thorough knowledge of exponential and linear functions. Knowledge of differential calculus is not required.

File organization

The task consists of linked TII files stored in the following folders:

- C:\TII\Growth contains the starting file and the teacher guide. Students start work by opening Start_Growth in TI InterActive!. This file contains hyperlinks to the tasks, which can be found in the folders C:\TII\Growth\linear, C:\TII\Growth\exponential, C:\TII\Growth\logistic and C:\TII\comparison.
- C:\TII\Growth\video_animation_help contains hyperlinks to video animations showing how specific tasks can be carried out in TI InterActive!. The videos are in the same folder and can be recognized by the extention scm, which stands for Screencam, the Lotus software that is used for the videos.

- C:\TII\Growth\linear contains the files concerning linear growth. The starting file for linear growth is linear_growth_Notebook, in which the students write their solutions. The folder contains hint files for mathematical assistance.
- C:\TII\Growth\exponential contains the files concerning exponential growth. The starting file for exponential growth is expo_growth_notebook, in which the students write their solutions. The folder contains hint files for mathematical assistance.
- C:\TII\Growth\logistic contains the files concerning logistic growth. The starting file is logi_growth_Notebook, in which the students write their solutions. The folder contains hint files for mathematical assistance.
- C:\TII\Growth\comparison contains the files which address the comparison of the three different growth models. The starting file here is comp_Notebook, in which the students write their solutions. The folder contains hint files for mathematical assistance.

The hyperlinks only work if the files are installed in the appropriate folders.

Suggestions for the teacher

The topic should be started with a general introduction and the presentation of introductory examples by the teacher (about 20 minutes). The teacher should also repeat the most relevant TII commands. If desired, the students can work in pairs. Students can explore the problem graphically with the help of sliders or by substituting values for the existing parameters. It is recommended that the teacher draws up an inventory of the students' findings and leads a classroom discussion on the issues that arise. The teacher may also present a correct solution or invite students to do so.

Technical hints

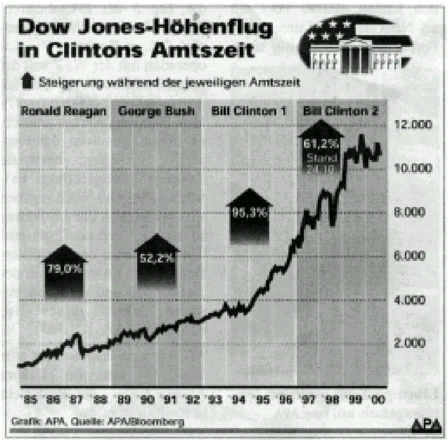
Students start by opening the file Start_Growth in TII. It is recommended that the different TII-files are opened in separate windows so that several versions of TII are activated. By means of a right mouse click in the Windows menu bar the option 'Windows Cascade' offers opportunities for switching easily between the files. However, if the same hyperlink is used twice two versions of the same file will be opened! This may be a source of confusion and therefore requires some attention by the teacher.

As TII only supports absolute hyperlinks and no relative links, the links need to be adjusted after installation of the files, for example on the school network.

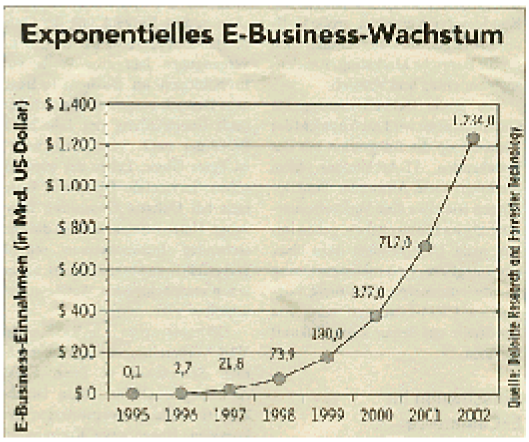
Examples

The following pictures show some examples of various kinds of growth. What kind of growth would one expect in these cases? In this unit you will develop models to describe the most important kinds of growth.

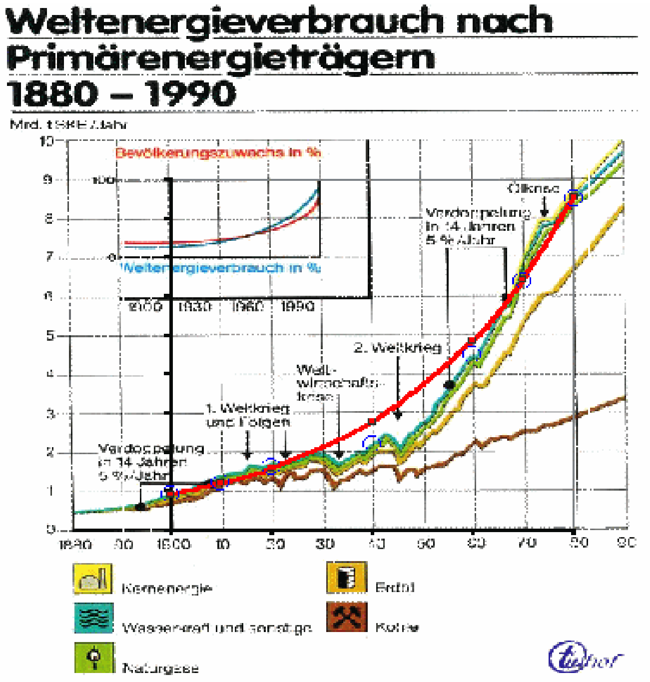
The Dow Jones Index



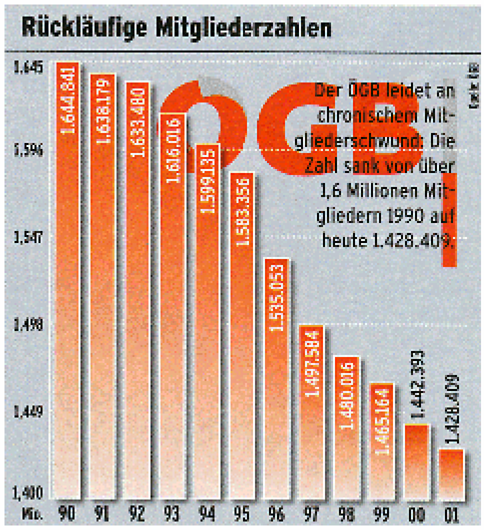
The Growth of e-business



The World Energy Consumption



The Number of Tradeunionists



Three types of mathematical growth models

$y(t)$	population at time t
$yo = y(0)$	population at the beginning of exploration
$\Delta y = y(t+1)-y(t)$	absolute increment
$\frac{\Delta y}{y} = \frac{y(t+1) - y(t)}{y(t)}$	relative increment; rate of growth

________________________________formula___

Linear growth (red)

$$y(t) = yo + t \cdot a$$

________________________________formula___

Exponential growth (blue)

$$y(t) = yo \cdot (1+i)^{t} \quad \text{or} \quad y(t) = yo \cdot e^{(k.t)}$$

________________________________formula___

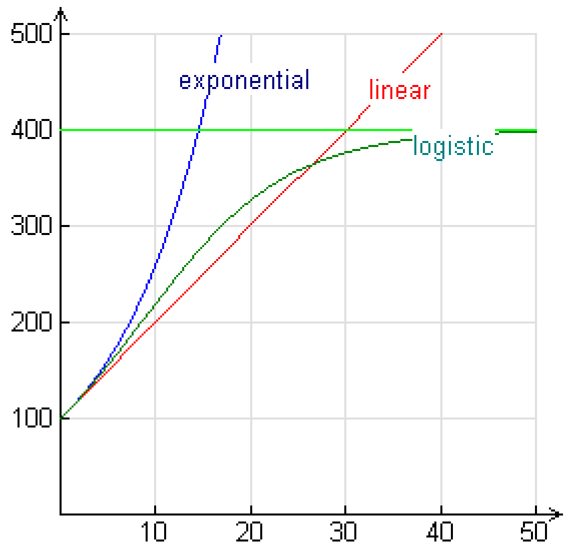
Logistic growth (green)

M maximum; saturation:

$$y(t) = \frac{M}{1 + b \cdot c^{t}} \quad \text{with} \quad b = \frac{M - yo}{yo}$$

or

$$y(t) = \frac{M}{1 + b \cdot e^{(-k.t)}} \quad \text{with} \quad b = \frac{M - yo}{yo}$$



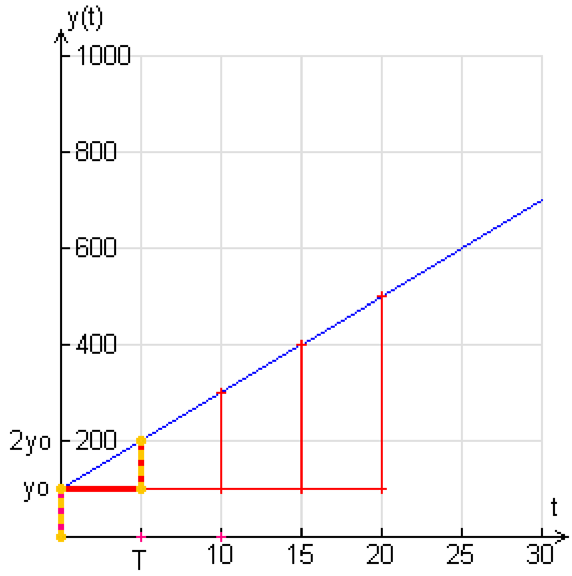
3.1 Linear growth

intro

We remember that the graphs of linear functions are defined by: *(linear growth)*

$y(t) = yo + a \cdot t$

On the right is the graph of the function for *yo=100* and *a = 20.*



task

Main task

Answer the following questions. **Hint 1**

a. Define $a = 20$.
What is the effect of the value of *yo* on the graph of the function?
Illustrate the answer using a Graph window.

b. Define yo =100.
What is the aeffect of the value of a on the graph of the function?
Illustrate the answer using a Graph window.

c. Define yo = 50, $a = 20$.
In what period T does the function value yo duplicate?

d. What happens to T if the value of yo is changed?
Calculate the values of T for yo = 50; 100; 150; 200; 300 and a = 20.

What happens to T if the value of a is changed?
Calculate the values of T for a = 10; 20; 30; 50; 100 and yo = 50.

task

Control task

a. Calculate a general equation for the double period T for linear growth. y(T) = 2*yo. **Hint 2**

b. Define yo = 90 and a =15.
Calculate the period in which the function value yo is triple the initial value .

c. Dora has 80.00 in a box.
At the end of each of the following weeks, she adds 20.00.
How much money is there after 5, 6, 7, t weeks in Doras box?

d. After how many months is the amount of Dora's money in the box 200.00?

3.2 Exponential growth

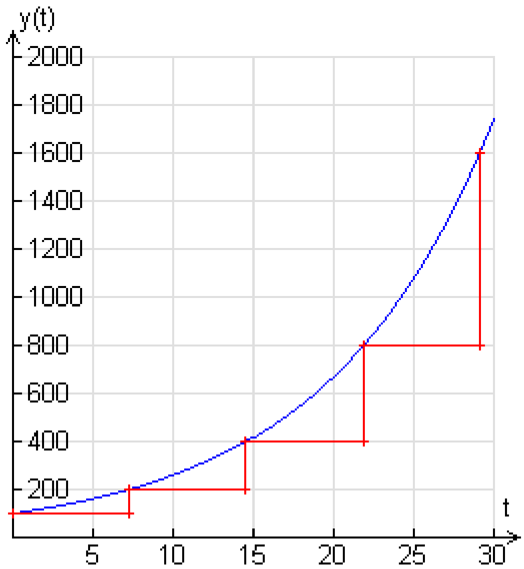
intro

We remember that the graph of exponential functions are defined by:

$y(t) = yo \cdot (1+i)^t$ *exponential growth*

On the right is the graph of the function for

yo=100 *and* i = 10% ; i = 0.1.



task

Main task A **Hint 3**

a. Define i = 0.1.
What is the influence of the value of yo on the graph of the function?
In what period T is the duplication of the function value y(t)?

b. Define yo =200.
What is the influence of the value of i on the graph of the function?
In what period T is the duplication of the function value y(t)?

c. Define yo = 100, i = 0.1.
In what period T is the duplication of the function value y(t)?
What happens with T, if the value of yo is changed?

task

Control task A **Hint 4**

a. Calculate a general equation for the double period T.
Is T constant?
Does T depend on yo?
Does T depend on i ?

b. Define i = 20% and yo = 300.
Calculate the period for the function value to be trebled.

c. Define i = 20% and yo = 300.
In what period does the function value becomes n*yo.

task

Main task B **Hint 5**

a. y(3) = 25
y(4) = 31.5
Calculate the absolute increment.
Calculate the relative increment.

b. y(2) = 25
y(7) = 31.5
Calculate the absolute increment.
Calculate the relative increment.

c. Define yo = 300, i = 0.1.

Plot the graph of a function deltay(t) $[\text{deltay}(t) := y(t) - y(t\text{-}1)]$ for the absolute increment of the exponential growth.

Plot the graph of a function rely(t) $\left[\text{rely}(t) := \frac{y(t) - y(t\text{-}1)}{y(t\text{-}1)}\right]$ for the relative increment of the exponential growth.

Describe both graphs and discribe the difference between the two increment functions.

d. Change the value of yo.
What is the influence of this value on the two increment functions?

e. Change the value of i.
What is the influence of this value on the two increment functions?

task

Control task B

Hint 6

a. yo = 0.9
y(80) = 8.5
Calculate the absolute increment and the relative rate of growth for 80 years.

b. yo = 0.9
y(80) = 8.5
Calculate the relative rate of growth i for *one* period.
Write the equation of exponential growth and plot the function.

task

Additional task

Hint 7

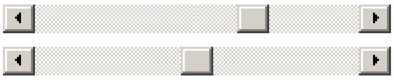
C-14 is a radioactive isotope with a half-life period of T = 5730 (+/- 1%) years.

a. How old is a skeleton with 10% of the native C-14 content?

b. Calculate an interval for the age of the skeleton.

Comparison between exponential and linear growth

Suppose in the first year both kinds of growth have the same increment.



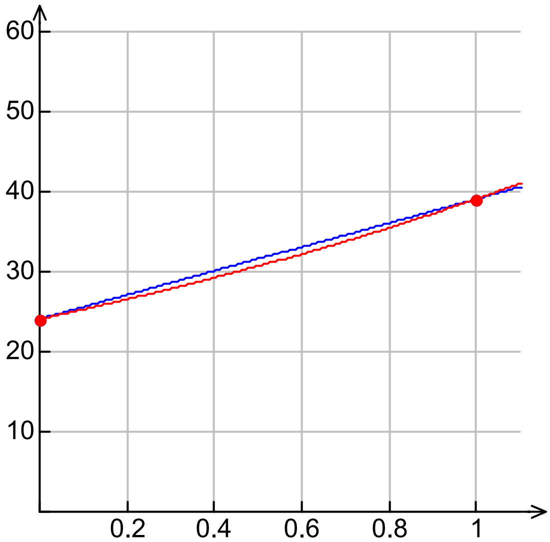
$yo = 24.1$

$a = 15$

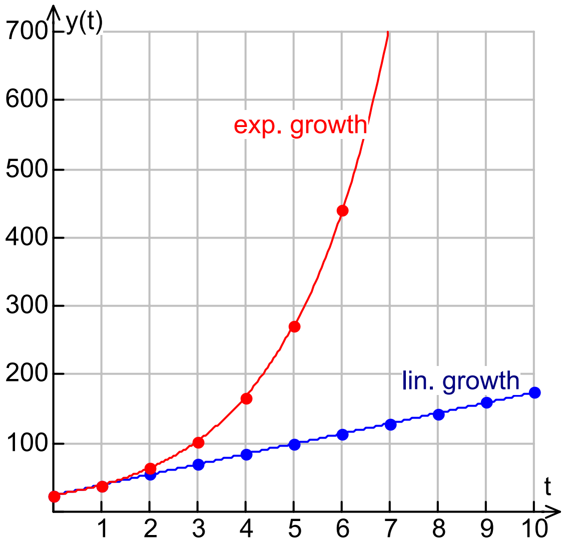
$i := \frac{a}{yo} = .622406639$

$\text{seq}(a,a,0,20) \rightarrow L1$ $yo + 11 \cdot a \rightarrow L2$ $yo + (1+i)^{11} \rightarrow L3$

Interval [0 ; 1]



Interval [0 ; 10]



3.3 Logistic growth

intro

We recall that logistic functions are defined by:

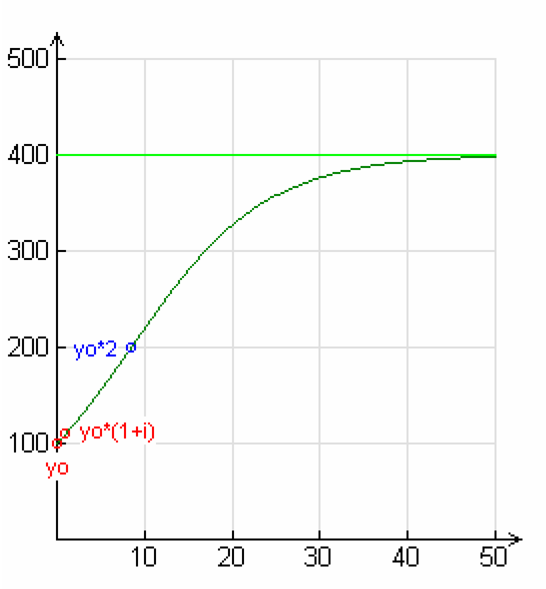
$$y(t) := \frac{M}{1 + b \cdot c^t} \quad \text{or} \quad y(t) = \frac{M}{1 + b \cdot e^{(-k \cdot t)}} ,$$

with $b := \frac{M - yo}{yo}$.

On the right you see the graph of the function for

yo=100 and i = 10% = 0.1.

Rate of growth in the first period: M = 400.



task

Main task A

Hint 8

a. Define yo =100, i = 0.1 and M = 400 with sliderbars.
Calculate b and c and graph the function.
What is the influence of the value of yo on the graph of the function?

b. Define yo =100, i =0.1 and M with a slider.
What is the influence of the value of M on the graph of the function?

c. Define yo =100, i and M = 400 with a slider.
What is the influence of the value of i on the graph of the function?

d. Define yo, i =0.1 and M = 400.
At what time T does the duplication of the function value yo occur?
What happens to T if the value of yo is changed?

task

Control task A

Hint 9

a. Define i = 15%, yo = 90 and M = 450 in the logistic growth model.
Find the time for the function value to become double the initial value yo.

b. Define i = 15%, yo = 50 and M = 450.
Find the time when the value of the function is triple the initial value of yo.

task

Main task B

Hint 10

a. Define yo = 100, y(1) = 130 and M = 400.

Plot a function deltay(t) $[\text{deltay}(t) := y(t) - y(t-1)]$ for the *absolute increment* of the logistic growth.

Plot a function rely(t) $\left[\text{rely}(t) := \frac{y(t) - y(t-1)}{y(t-1)}\right]$ for the *relative increment* of the logistic growth.

Describe the function of the absolute increment.

b. Change the value of y(1).
What is the influence of this value on the function deltay(t)?

task

Control task B

Hint 11

Suppose yo = 15, y(4) = 23 and M = 50.

a. Calculate the formula for logistice growth function.

b. Plot the graph of the logistic function and the function with absolute increment.

c. Calculate the maximum of the absolute increment.

task

Additional task

Hint 12

In 1980 4.5 billion people lived on the Earth.
In 1999 6 billion lived on our planet.

a. One model of growth predicts an exponential growth for the number of people on Earth.
Calculate the exponential growth function which fits the data.

b. Another growth model assumes logistic growth with a maximum of 15 billion people on Earth.
Calculate the logistic growth function with yo = 4.5, M = 15 and y(19) = 6.

c. Plot the graphs of both functions on one graph. Which model is more realistic, do you think?

3.4 Comparison

intro

We remember that the graphs of functions for the most important types of growth are defined by:

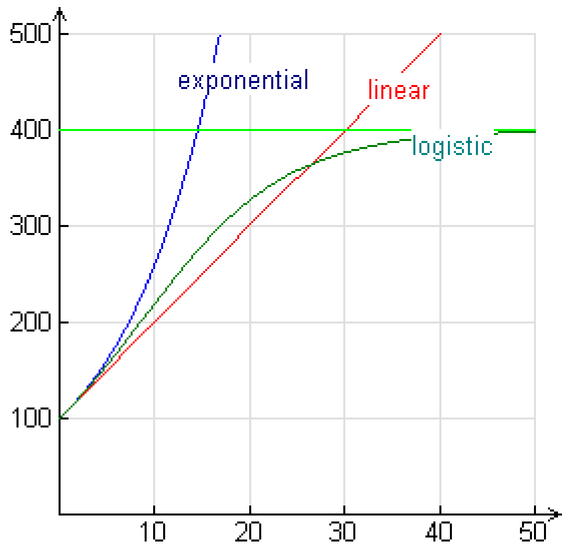
$\text{lin}(t) := yo + a \cdot t$ linear growth

$\text{ex}(t) := yo \cdot (1+i)^t$ exponential growth

$\text{logi}(t) := \dfrac{M}{1 + b \cdot c^t}$ logistic growth

$b := \dfrac{M - yo}{yo}$

On the right you see the graphs of the three functions for: yo=100, i = 10%, a = 10 and M = 400.



task

Main task

a. Define i = 0.1, a = i*yo, M = 400. **Hint 13**
What is the influence of the value of yo on the three functions?

b. Define yo =100, a = i*yo, M = 400. **Hint 14**
What is the influence of the value of i on the three functions?

c. Define yo = 100, i = 0.1, a = i*yo. **Hint 15**
What is the influence of the value of M on the three functions?

task

Control task **Hint 16**

a. Define M = 100, i = -10% and yo = 500.
Calculate the terms of the linear, exponential and the logistic function.

b. Explain what happens with the growth functions (exponential and logistic) when the value of i is negative and yo >M.

3.5 Hints

hint 1

a/b Define the formula of linear growth in a new MathBox: y(t):=yo+a*t.

Use slider bars to choose values for yo and a.

yo population yo at the start of exploration

a absolute increment of a

Show the graph of the function.

To calculate the double period (for the first period) use: solve(y(t)=2*yo,t).

$\text{solve}(y(t) = 2\cdot yo, t)$ double period T = Th; y(T)= 2*yo

c. Delete the values of yo and a for the following calculations: $\text{delvar}(yo) :: \text{delvar}(a)$.
Use solve(y(t)=2*yo,t) to calculate the double period T. Store the result as Th.

Use Th | yo=100 and a=20 to define yo = 100, a = 20 and to calculate Th.
$Th \mid yo = 100 \text{ and } a = 20$.

d. Use Th | yo=50 and a=20 to define yo = 100, a = 20 and to calculate Th.
For the second assignment in task D, use Th | yo=50 and a=10 to define yo = 100, a = 20 and to calculate Th.

hint 2

a. Define the formula of linear growth $y(t) := yo + a\cdot t$.
Use solve(y(t)=2*yo,t) to calculate t.

b. Use solve(y(t)=n*yo,t) and the "|" to calculate the periods.

c. Define y(t):=yo+a*t with yo and a.
Use a table to calculate the values of y(t).

d. Use solve(y(t)=20,t) to find the time needed to get a value of 20.

hint 3

a. Define the formula for exponential growth: $y(t) := yo\cdot(1+i)^{t}$.

Insert slider bars to define the values of yo and i:

population at the beginning of exploration

yo

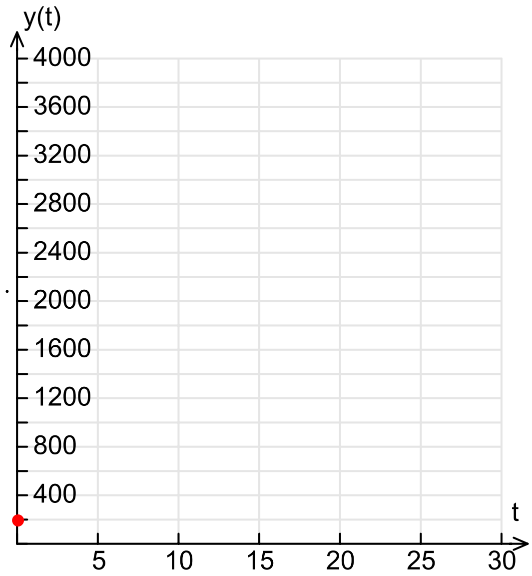
relative growth rate

i

Plot the graph of the function y(t) to answer the questions.

Calculation of doubling period T.

use: $\text{solve}(y(t)=2\cdot yo, t)$
store the right side of the answer as Th:
$\text{right}(ans) \rightarrow Th$



Calculation of doubling period T: y(T)= 2*yo.

Delete variable yo and a for further calculations:
$\text{delvar}(yo) :: \text{delvar}(i)$.

doubling period T:
$\text{solve}(y(t)=2\cdot yo, t)$
use "| yo=100 and i=0.1"
$\text{solve}(y(t)=2\cdot yo, t) \mid yo=100 \text{ and } i=0.1$

trebling period:
$\text{solve}(y(t)=3\cdot yo, t)$

factor n period:
$\text{solve}(y(t)=n\cdot yo, t)$
$\text{solve}(y(t)=n\cdot yo, t) \mid yo=100 \text{ and } i=0.1$

b. Give an answer like T1(i) =.... (You must *not* use the function name T(i)).
Use: $\text{solve}(y(t)=2\cdot yo, t)$.

c. Use $\text{solve}(y(t)=2\cdot yo, t)$ for the doubling period T
and $\text{solve}(y(t)=2\cdot yo, t) \mid yo=100 \text{ and } i=0.1$

hint 4

---

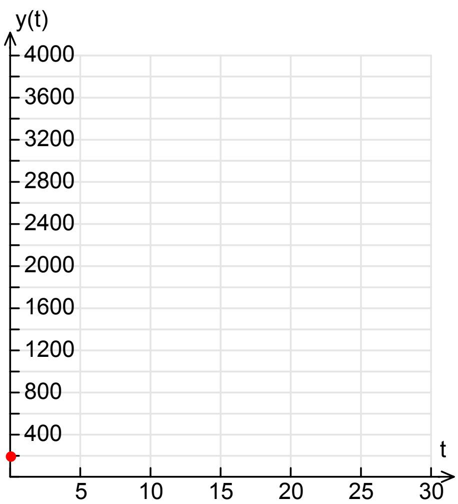
a. Define the formula for exponential growth: $y(t) := yo\cdot(1+i)^t$.
Insert slider bars to define the values of yo and i:

population at the start of exploration

 *yo*

relative growth rate

 *i*



Print the graph of the function y(t) to answer the questions.

Calculation of doubling period T: y(T)= 2*yo.

Calculation of double period T:

use: $\text{solve}(y(t)=2\cdot yo, t)$
store the right side of the answer as Th:
$\text{right}(ans) \rightarrow Th$

delete variable yo and a
$\text{delvar}(yo) :: \text{delvar}(i)$

b. trebling period:

$\text{solve}(y(t)=3\cdot yo, t)$

d. factor n period:

$\text{solve}(y(t)=n\cdot yo, t)$
$\text{solve}(y(t)=n\cdot yo, t) \mid yo=100 \text{ and } i=0.1$

a. y(3) = 25
y(4) = 31.5

absolute increment: $y(4) - y(3)$

relative increment: $\frac{y(4) - y(3)}{y(3)}$

relative increment in %: $\frac{y(4) - y(3)}{y(3)} \cdot 100\ \%$

$\frac{\Delta y}{\Delta t}$ $\frac{y(4) - y(3)}{1}$

b. y(2) = 25
y(7) = 31.5

absolute increment: $y(7) - y(2)$

relative increment: $\frac{y(7) - y(2)}{y(2)}$

relative increment in %: $\frac{y(7) - y(2)}{y(2)} \cdot 100\ \%$

$\frac{\Delta y}{\Delta t}$ $\frac{y(7) - y(2)}{1}$

$y(t) := yo \cdot (1 + i)^t$ Define the formula for exponential growth.
$yo := 300$ Define the population at the begining of exploration.
$i := 0.1$ Define the relative growth rate.

c. Define the formula of the absolute increment:
$deltay(t) := y(t) - y(t - 1)$.

Enter *deltay(t)* in a new Math Box to show the formula of the function.

Notice the exponential growth of the absolute increment!

Define the formula of the relative increment:

$rely(t) := \frac{y(t) - y(t - 1)}{y(t - 1)}$ relative increment

Enter *rely(t)* in a new Math Box to show the formula of the function.

Define the formula of the relative increment in %:

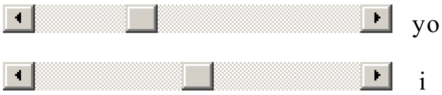
$rely(t) := \frac{deltay(t)}{y(t - 1)} \cdot 100$ relative increment in %

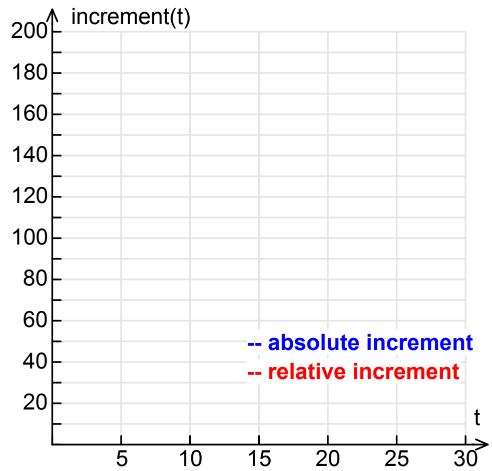
Enter *rely(t)* in a new Math Box to show the formula of the function.

$rely(t)\ \%$ relative increment in %

d/e. With a slider bar you can change the values of yo and i.

Watch the influence of the changes on the graph.





hint 6

a. Define the formula for exponential growth.

$y(t) := yo \cdot (1+i)^t$
$yo := 0.9$
$y80 := 8.5$

Calculate the absolute increment and the relative rate of growth for 80 years.

$y80 - yo \; = \; 8.5 - yo$	absolute increment
$\dfrac{y80 - yo}{yo}$	relative increment
$\dfrac{y80 - yo}{yo} \cdot 100 \quad \%$	relative increment in %

b. Open a new MathBox and solve the equation y(80)=8.5 with i >0:

$\text{solve}(y(80) = 8.5, i) \mid i > 0.$

Write in the next MathBox: $\text{right}(ans) \rightarrow i$.

Plot the graph of the function y(t).

hint 7

Define y(t) = yo*(1+i)^t and yo =100.

$y(t) := yo \cdot (1+i)^t$
$yo := 100$

Define Th=5730.

$Th := 5730$

Solve the equation y(Th)=yo/2 with i >-1.

$\text{solve}\left(y(Th) = \dfrac{yo}{2}, i\right) \mid i > -1$
$\text{right}(ans) \rightarrow i$
$y(t)$

Solve the equation y(t)=10.

Define ya(t) = yo*(1+i1)^t.

$\text{ya}(t) := yo \cdot (1 + i1)^t$
$Ta := Th \cdot 1.01$

Calculation see above!

Calculation with T= 5730 +1% = 5787.3 years.

Define ya(t) = yo*(1+i1)^t

$\text{ya}(t) := yo \cdot (1 + i1)^t$
$Ta := Th \cdot 1.01$

Calculation see above!

hint 8

a. Define yo, M and i with slider bars. Answer the questions using the sliders.

 yo

 M Maximum, saturation of logistic growth.

 i This means $i \cdot 100\%$ growth in the first time interval.

$y(t) := \dfrac{M}{1 + b \cdot c^t}$ Define the function of logistic growth y(t).

Important: Start with the calculation of b, using the solve command.
Then continue with the calculation of c.

b. Graph the function and use the slider bar to change the value of M. Examine the effects.

c. Graph the function and use the slider bar to change the value of i. Examine the effects.

d. If T is the duplication time, y(T)=2*y(0).

hint 9

a. Define yo, M and i.

Define y(t): $y(t) := \frac{M}{1 + b \cdot c^{t}}$.

First calculate b and c. If the initial value yo is doubled, y(t) = 2*yo.

b. Follow the approach of the previous assignment, but this time notice that y(t) = 3*yo.

hint 10

a. First calculate the value of c using the given value of y(1).

b. Change the value of y(1) and examine the effect this has on the graph of deltay.

hint 11

a. Define yo and M.

Define y(t): $y(t) := \frac{M}{1 + b \cdot c^{t}}$.

Use yo to calculate the value of b.
Use y(4) to calculate the value of c.

b. The absolute increment function is defined by $\text{deltay}(t) := y(t) - y(t-1)$.

c. Use Calculate Maximum for finding the maximum of the absolute increment.

hint 12

a. The general model for exponential growth is $\text{ex}(t) := yo \cdot (1+i)^{t}$.
Use the data to calculate yo and i.

b. The general model for logistic growth is $y(t) := \frac{M}{1 + b \cdot c^{t}}$.

Use the data points and M=15 to calculate b and c.

hint 13

First define yo with a slider and i, M and a with:

yo

$i := 0.1$
$M := 400$
$a := i \cdot yo$

Then enter the three growth models.

hint 14

First define i with a slider and yo, a and M with:

 i

$yo := 100$
$M := 400$
$a := i \cdot yo$

Then enter the three growth models.

hint 15

First define M with a slider and yo, i and a with:

 M

$yo := 100$
$i := 0.1$
$a := i \cdot yo$

Then enter the three growth models.

hint 16

a. Start with an example:
$yo := 500$
$i := -0.1$
$M := 300$

Find the formulas for the three types of growth.

b. Use the above example to investigate the three graphs.